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Inextensible Curve Flows According to Bishop 2-Type Frame in Minkowski 3-Space

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Abstract

In this paper, we derive necessary and sufficient conditions for inextensible curve flow of timelike curves and spacelike curves with timelike normal according to Bishop 2-type frame in the Minkowski-3 space R_1^3.

Keywords: Bishop 2-type, Minkowski 3-space, inextensible

1 Introduction

The flow of a curve is called to be inextensible if its arclength is preserved [5]. Kwon, Park and Chi investigated inextensible flows of curves and developable surfaces in Euclidean 3-space [6]. Gürbüz study inextensible curve flows for null and non-null curves in Euclidean 3-space [1-3]. Yıldız, Tosun and Karakuş investigated inextensible flows of curves in E^n [10]. New version of Bishop frame was studied by Yılmaz [8]. In [9], authors studied characterizations of non-null curves according to the Bishop frame of type-2 in Minkowski 3-space. Kızıltuğ was investigated characterization of inextensible flows of curves according to Type-2 Bishop frame Euclidean 3-space [4]. In this paper we study inextensible non-null curve flows according to Bishop 2-type frame in Minkowski 3-space.

Minkowski 3-space $\mathcal{R}_1^3$ is given the following metric [7] :

$$\langle , \rangle_M = -dx_1^2 + dx_2^2 + dx_3^2$$

Frenet-Serret frame $\{\mathcal{T}, \mathcal{N}, \mathcal{B}\}$ derivative formulas are given in $\mathcal{R}_1^3$ as following:

$$\begin{aligned} \mathcal{T}_s &= \epsilon_2 \kappa \mathcal{N}, \\ \mathcal{N}_s &= -\epsilon_1 \kappa \mathcal{T} + \epsilon_3 \tau B, \\ B_s &= -\epsilon_2 \tau \mathcal{N}. \end{aligned} \tag{1}$$

Here κ, τ Frenet curvatures with

$$\langle \mathcal{T}, \mathcal{T} \rangle_M = \epsilon_1, \langle \mathcal{N}, \mathcal{N} \rangle_M = \epsilon_2, \langle B, B \rangle_M = \epsilon_3, \epsilon_i = \pm 1$$

For an arbitrary vector $x = (x_1, x_2, x_3)$ in $\mathcal{R}_1^3$, if $\langle x, x \rangle_M > 0$, x is spacelike, if $\langle x, x \rangle_M < 0$, x is timelike. Spacelike and timelike vectors are called non-null vectors.

Let $\{V_1, V_2, B\}$ be Bishop2-type frame with

$$\langle V_1, V_1 \rangle_M = \varepsilon_1, \langle V_2, V_2 \rangle_M = \varepsilon_2, \langle B, B \rangle_M = \varepsilon_3.$$

Bishop 2-type frame derivative formulas are presented for timelike curves and spacelike curves with timelike normal in $\mathcal{R}_1^3$ as different from [9] in [3] as following :

Theorem 1.1 [3] Let $\{\mathcal{T}, \mathcal{N}, B\}$ be Frenet frame for a spacelike curve with timelike normal

$$\langle \mathcal{T}, \mathcal{T} \rangle_M = 1, \langle \mathcal{N}, \mathcal{N} \rangle_M = -1, \langle B, B \rangle_M = 1$$

and let $\{V_1, V_2, B\}$ be Bishop 2-type frame with

$$\langle V_1, V_1 \rangle_M = 1, \langle V_2, V_2 \rangle_M = -1, \langle B, B \rangle_M = 1$$

Bishop 2-type frame derivative formulas are given by as following in $\mathcal{R}_1^3$:

$$\begin{bmatrix} V_{1s} \\ V_{2s} \\ B_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & -k_1 \\ 0 & 0 & k_2 \\ k_1 & k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ B \end{bmatrix} \tag{2}$$

where k_1, k_2 are curvatures according to Bishop 2-type frame in the Minkowski 3-space. Connection between Frenet frame and Bishop 2-type frame is expressed as following:

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ B \end{bmatrix}$$

First and second Bishop 2-type curvatures are $k_1 = -\langle V_{1s}, B \rangle_M = \tau \sinh\theta$, $k_2 = \langle V_{2s}, B \rangle_M = \tau \cosh\theta$. Also $\theta_s = -\kappa$, $\tau = |k_1^2 - k_2^2|^{1/2}$.

Proof . The tangent vector $\mathcal{T}$ can be written by

$$\mathcal{T} = \cosh\theta V_1 + \sinh\theta V_2 \tag{3}$$

Taking derivative of (3), substituting $V_{1s} = -k_1 B$, $V_{2s} = k_2 B$ we obtain

$$\mathcal{N} = \sinh\theta V_1 + \cosh\theta V_2, \quad \theta = \arg\tanh\frac{k_1}{k_2},\ \theta_s = -\kappa$$

From derivative of binormal,

$$B_s = k_1 V_1 + k_2 V_2 = \tau\mathcal{N} \tag{4}$$

Taking norm of (4), we have $\tau = |k_1^2 - k_2^2|^{1/2}$.

Theorem 1.2 [3] Let $\{\mathcal{T}, \mathcal{N}, B\}$ be Frenet frame with $\langle\mathcal{T},\mathcal{T}\rangle_M = -1$, $\langle\mathcal{N},\mathcal{N}\rangle_M = 1$, $\langle B, B\rangle_M = 1$ and let $\{V_1, V_2, B\}$ be Bishop 2-type frame with $\langle V_1, V_1\rangle_M = -1$, $\langle V_2, V_2\rangle_M = 1, \langle B, B\rangle_M = 1$. Bishop 2-type frame derivative formulas for timelike curves are given as following :

$$\begin{bmatrix} V_{1s} \\ V_{2s} \\ B_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & -k_1 \\ 0 & 0 & k_2 \\ -k_1 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ B \end{bmatrix}$$

where k_1, k_2 are curvatures according to Bishop 2-type frame in Minkowski 3-space. Connection between Frenet frame and Bishop 2-type frame is expressed as following:

$$\begin{aligned} \mathcal{T} &= \cosh\theta V_1 + \sinh\theta V_2 \\ \mathcal{N} &= \sinh\theta V_1 + \cosh\theta V_2 \\ B &= B \end{aligned}$$

First and second Bishop 2-type curvatures are $k_1 = \tau\sinh\theta$, $k_2 = \tau\cosh\theta$. Here $\theta_s = \kappa$. Proof is obtained as similar with Theorem 1.1. As result, Bishop 2-type frame derivative formulas for timelike curves and spacelike curves with timelike normal are given as following [3]:

$$\begin{bmatrix} V_{1s} \\ V_{2s} \\ B_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & -k_1 \\ 0 & 0 & k_2 \\ \varepsilon_1 k_1 & \varepsilon_1 k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ B \end{bmatrix} \tag{5}$$

where

$$\langle V_1, V_1\rangle_M = \varepsilon_1, \langle V_2, V_2\rangle_M = \varepsilon_2, \langle B, B\rangle_M = \varepsilon_3, \varepsilon_i = \pm 1$$

Here k_1, k_2 are first and second Bishop 2-type curvatures in Minkowski 3-space.

2 Inextensible flows of non-null curves according to Bishop 2-type frame in Minkowski 3-space

We consider $P(\xi,t):[0,l]\times[0,t]\to\mathcal{R}_1^3$ one parameter family. Flow of spacelike curves with timelike normal and timelike curves according to Bishop 2-type frame is expressed as following

$$\frac{\partial P(\xi,t)}{\partial t}=fV_1+gV_2+hB$$

where f,g,h are components of flow of non null curve family. Let

$$v=\left|\left\langle\frac{\partial P}{\partial\xi},\frac{\partial P}{\partial\xi}\right\rangle_M\right|^{1/2} \tag{6}$$

be curve speed and arc length is $s=\int\limits_0^\xi v d\xi$ according to Bishop 2-type frame.

Definition 2.1. $\frac{P(\xi,t)}{\partial t}$ are defined to be inextensible according to Bishop 2-type frame if $\frac{\partial}{\partial t}\sqrt{\left|\left\langle\frac{\partial P}{\partial\xi},\frac{\partial P}{\partial\xi}\right\rangle_M\right|}=0$

Lemma 2.1.

$$\frac{\partial v}{\partial t}=\varepsilon_1\cosh\theta(\frac{\partial f}{\partial t}+vhk_1)+\varepsilon_2\sinh\theta(\frac{\partial g}{\partial t}+vhk_2) \tag{7}$$

Proof. From (6), it is obtained

$$2v\frac{\partial v}{\partial t}=\frac{\partial}{\partial t}\left|\left\langle\frac{\partial P}{\partial\xi},\frac{\partial P}{\partial\xi}\right\rangle_M\right|=2v\left|\left\langle\cosh\theta V_1+\sinh\theta V_2,\frac{\partial}{\partial t}(fV_1+gV_2+hB)\right\rangle_M\right| \tag{8}$$

From (8), we obtain (7).

Theorem 2.1. Spacelike curve with timelike normal and timelike curve flow $\frac{\partial P}{\partial t}=fV_1+gV_2+hB$ according to Bishop 2-type frame is inextensible if and only if

$$\varepsilon_1\cosh\theta(\frac{\partial f}{\partial t}+vhk_1)=-\varepsilon_2\sinh\theta(\frac{\partial g}{\partial t}+vhk_2). \tag{9}$$

Proof. If spacelike curve with timelike normal and timelike curve not be subject to any elongation or compression, the condition

$$\frac{\partial}{\partial t}s(\xi,t)=\int\limits_0^\xi\frac{\partial v}{\partial t}d\xi=0$$

is satisfied. From Lemma 2.1,

$$\frac{\partial}{\partial t}s(\xi,t)=\int_0^{\xi}\left|\varepsilon_1\cosh\theta(\frac{\partial f}{\partial t}+vhk_1)+\varepsilon_2\sinh\theta(\frac{\partial g}{\partial t}+vhk_2)\right|d\xi=0 \tag{10}$$

From (10) we obtain (9).

Theorem 2.2.

$$\frac{\partial V_1}{\partial t}=[-\varepsilon_2\sinh\theta(\varepsilon_1(\frac{\partial f}{\partial s}+\varepsilon_1hk_1)-\varepsilon_1\sinh\theta\frac{\partial\theta}{\partial t})-\varepsilon_2\Psi_1]V_2-\varepsilon_3\Phi_1B, \tag{11}$$

$$\frac{\partial V_2}{\partial t}=[-\varepsilon_1\cosh\theta(\varepsilon_2(\frac{\partial g}{\partial s}+\varepsilon_1hk_2)-\varepsilon_2\cosh\theta\frac{\partial\theta}{\partial t})+\varepsilon_1\Psi_1]V_1-\varepsilon_3\Phi_2B, \tag{12}$$

$$\frac{\partial B}{\partial t}=V_1[\varepsilon_1\Phi-\varepsilon_3(\frac{\partial h}{\partial s}-fk_1+gk_2)\cosh\theta]+V_2[\varepsilon_2\Phi_2-\varepsilon_3(\frac{\partial h}{\partial s}-fk_1+gk_2)\sinh\theta]. \tag{13}$$

Proof.

$$\frac{\partial\mathcal{T}}{\partial t}=\frac{\partial}{\partial s}\frac{\partial P}{\partial t}=(\frac{\partial f}{\partial s}+\varepsilon_1hk_1)V_1+(\frac{\partial g}{\partial s}+\varepsilon_1hk_2)V_2+(\frac{\partial h}{\partial s}-fk_1+gk_2)B \tag{14}$$

From (14),

$$\left\langle\frac{\partial\mathcal{T}}{\partial t},V_1\right\rangle_M=\varepsilon_1(\frac{\partial f}{\partial s}+\varepsilon_1hk_1). \tag{15}$$

Also

$$\left\langle\frac{\partial\mathcal{T}}{\partial t},V_1\right\rangle_M=\sinh\theta(\varepsilon_1\frac{\partial\theta}{\partial t}+\left\langle\frac{\partial V_2}{\partial t},V_1\right\rangle_M) \tag{16}$$

From (15), (16) we obtain

$$\sinh\theta(\varepsilon_1\frac{\partial\theta}{\partial t}+\left\langle\frac{\partial V_2}{\partial t},V_1\right\rangle_M)=\varepsilon_1(\frac{\partial f}{\partial s}+\varepsilon_1hk_1). \tag{17}$$

$$\left\langle\frac{\partial V_1}{\partial t},V_2\right\rangle_M+\left\langle V_1,\frac{\partial V_2}{\partial t}\right\rangle_M=0\Rightarrow\frac{\partial V_1}{\partial t}=-\varepsilon_2\Psi_1V_2, \tag{18}$$

$$\left\langle\frac{\partial V_1}{\partial t},B\right\rangle_M+\left\langle V_1,\frac{\partial B}{\partial t}\right\rangle_M=0\Rightarrow\frac{\partial V_1}{\partial t}=-\varepsilon_3\Phi_1B, \tag{19}$$

where

$$\Psi_1=\left\langle V_1,\frac{\partial V_2}{\partial t}\right\rangle_M,\ \Phi_1=\left\langle V_1,\frac{\partial B}{\partial t}\right\rangle_M. \tag{20}$$

With aid (17), (18), (19) we obtain (11). From (14)

$$\left\langle \frac{\partial \mathcal{T}}{\partial t}, V_2 \right\rangle_M = \varepsilon_2 (\frac{\partial g}{\partial s} + \varepsilon_1 h k_2), \tag{21}$$

$$\left\langle \frac{\partial \mathcal{T}}{\partial t}, V_2 \right\rangle_M = \cosh\theta (\left\langle \frac{\partial V_1}{\partial t}, V_2 \right\rangle_M + \varepsilon_2 \frac{\partial \theta}{\partial t}). \tag{22}$$

From (22),

$$-\cosh\theta (\left\langle \frac{\partial V_2}{\partial t}, V_1 \right\rangle_M + \varepsilon_2 \frac{\partial \theta}{\partial t}) = \varepsilon_2 (\frac{\partial g}{\partial s} + \varepsilon_1 h k_2) \tag{23}$$

From (20),

$$\frac{\partial V_2}{\partial t} = \varepsilon_1 \Psi_1 V_1, \tag{24}$$

$$\left\langle \frac{\partial V_2}{\partial t}, B \right\rangle_M + \left\langle V_2, \frac{\partial B}{\partial t} \right\rangle_M = 0 \Rightarrow \frac{\partial V_2}{\partial t} = -\varepsilon_3 \Phi_2 B, \tag{25}$$

where

$$\left\langle V_2, \frac{\partial B}{\partial t} \right\rangle_M = \Phi_2 \tag{26}$$

With aid (23),(24), (25) we obtain (12)

$$\left\langle \frac{\partial \mathcal{T}}{\partial t}, B \right\rangle_M = \varepsilon_3 (\frac{\partial h}{\partial s} - f k_1 + g k_2), \tag{27}$$

$$\left\langle \frac{\partial \mathcal{T}}{\partial t}, B \right\rangle_M = \cosh\theta \left\langle \frac{\partial V_1}{\partial t}, B \right\rangle_M + \sinh\theta \left\langle \frac{\partial V_2}{\partial t}, B \right\rangle_M \tag{28}$$

From (26), (27), (28) we have (13).

Theorem 2.3. Assume spacelike curve with timelike normal flow and timelike curve flow $\frac{\partial P}{\partial t}$ is inextensible for Bishop 2-type . Necessary and sufficient conditions for inextensible non-null curve flow are given as a system of partial differential equation contain first and second Bishop 2-type curvatures as following:

$$\frac{\partial k_1}{\partial t} = k_2 [\varepsilon_2 \Psi_1 + \varepsilon_2 \sinh\theta (\varepsilon_1 (\frac{\partial f}{\partial s} + \varepsilon_1 h k_1) - \varepsilon_1 \sinh\theta \frac{\partial \theta}{\partial t})] + \varepsilon_3 \frac{\partial \Phi_1}{\partial s} \tag{29}$$

$$\frac{\partial k_2}{\partial t} = -\varepsilon_3 \frac{\partial \Phi_2}{\partial s} - k_1 [(\varepsilon_1 \Psi_1 - \varepsilon_1 \cosh\theta (\varepsilon_2 (\frac{\partial g}{\partial s} + \varepsilon_1 h k_2) - \varepsilon_2 \cosh\theta \frac{\partial \theta}{\partial t})] \tag{30}$$

Proof.

$$\frac{\partial}{\partial t} \frac{\partial V_1}{\partial s} = \frac{\partial}{\partial t} (-k_1 B) = -\frac{\partial k_1}{\partial t} B - k_1 \frac{\partial B}{\partial t}, \tag{31}$$

$$\frac{\partial}{\partial s}\frac{\partial V_1}{\partial t} = [k_2\Omega - \varepsilon_3\frac{\partial \Phi_1}{\partial s}]B + [\frac{\partial \Omega}{\partial s} - \varepsilon_3\varepsilon_1 k_2\Psi_1]V_2 - \varepsilon_1\varepsilon_3 k_1\Phi_1 V_1 \tag{32}$$

where

$$\Omega = -\varepsilon_2\Psi_1 + \varepsilon_1\varepsilon_2 \sinh\theta(\sinh\theta\frac{\partial\theta}{\partial t} - (\frac{\partial f}{\partial s} + hk_1)).$$

$$\frac{\partial}{\partial t}\frac{\partial V_1}{\partial s} = \frac{\partial}{\partial s}\frac{\partial V_1}{\partial t}$$

and using compability conditions (31), (32), we obtain (29). From

$$\frac{\partial}{\partial t}\frac{\partial V_2}{\partial s} = \frac{\partial}{\partial s}\frac{\partial V_2}{\partial t}$$

we obtain (30).

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