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Research Article

Generalized Elastic Line, Differentials of Maps Connected with CNLSE via Conformal Derivative and Magnetic Curves for Darboux Frame

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ABSTRACT

In this paper, we study generalized elastic lines via the intrinsic directional derivatives for the Darboux frame in $\mathbb{R}^3$. Subsequently, we investigate the congruences of surfaces associated with the Darboux frame in $\mathbb{R}^3$. Furthermore, we establish the Darboux conformal derivative equations and conformal magnetic curves for the Darboux frame in $\mathbb{R}^3$. Finally, we present formulas for conformal differentials of maps associated with the CNLSE for the Darboux frame in $\mathbb{R}^3$.

Keywords: Darboux frame, generalized elastic line, anholonomic coordinates, conformal magnetic curve
2020 Mathematics Subject Classification: 53A05, 53A35, 53A99

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1 Introduction

In recent years, the topic of integrable flows of curves has attracted increasing interest among researchers due to its profound relevance across various domains of physics and mathematics. This area has received considerable attention, particularly because of its applications in soliton theory, gauge field theory, nonlinear partial differential equations, as well as in geometric modeling, computer graphics, and image processing. Several studies have explored relativistic particle models including curvature and torsion to achieve a more comprehensive understanding of particle dynamics [10, 11, 22, 38, 40, 45, 46]. Hasimoto demonstrated that the motion of a thin, isolated vortex filament corresponds to the Nonlinear Schrödinger Equation (NLSE) via the Hasimoto map $\zeta = \frac{1}{\sqrt{2}}\kappa e^{i\int \tau d\mathfrak{T}}$ in $\mathbb{R}^3$ [35]. Langer and Perline presented the recursional, normalization, differential formulas of Hasimoto map for the Frenet frame in $\mathbb{R}^3$ [39]. Moreover, Ertuğ Gürbüz obtained formulas for differentials of Hasimoto-like maps with respect to various frames [19, 26, 30].

The investigation of the connection between optics and elastic lines has novel applications across diverse fields, from telecommunications to biotechnology including the development of innovative photonic devices, adaptive optical systems, and optomechanical sensors. The relationship between vortex filament motion and elastica was first introduced by Hasimoto [34]. Recently, many authors have studied variational problems related to elasticity [3, 4, 8, 10, 17, 22–24, 27–29, 31–33, 41, 43, 48]. This topic delves into the practical applications of models within the domain of quantum field theory and their relevance in analyzing solutions associated with black holes.

Darboux frame derivative equations with the unit tangent vector $\mathbf{T}$, the unit normal of surface $\mathbf{Z}$, the unit vector $\mathbf{Q} = \mathbf{T} \times \mathbf{Z}$, the geodesic curvature κ_g, geodesic torsion τ_g and the normal curvature κ_n in the direction $\mathfrak{T}-lines$ in $\mathbb{R}^3$ are given by $\mathbf{T}_{\mathcal{T}} = \kappa_g\mathbf{Q}+\kappa_n\mathbf{Z}$, $\mathbf{Q}_{\mathcal{T}} = -\kappa_g\mathbf{T}+\tau_g\mathbf{Z}$, $\mathbf{Z}_{\mathcal{T}} = -\kappa_n\mathbf{T} - \tau_g\mathbf{Q}$ [12]. Darboux frame is utilized to analyze the intrinsic geometry of curves and surfaces, and has a crucial role in quantum physics and optics for analyzing the propagation of optical waves, such as diffraction, waveguides, lasers, and various aspects of quantum mechanics. Moreover, it is used in deriving analytical solutions in quantum mechanics, such as the NLSE. The connection between Darboux frame and changes in the electric field is crucial for understanding the connection between geometric structures and physical phenomena [6, 14–17]. Intrinsic directional derivatives in anholonomic coordinates have been studied by various authors [17, 18, 20, 21, 42, 47].

Fractional calculus is an extension of conventional calculus, which involves the differentiation and integration of non-integer orders. The conformal derivative of the function f of the order α is $D^{\alpha}f(x) = \lim_{h \to 0} \frac{f(x+hx^{1-\alpha})-f(x)}{h}$ and $\alpha-$fractional integral of the function f, starting from x is $\mathbb{I}_{\alpha}^{x}(f) = \int_{x}^{t} \frac{f(a)}{a^{1-\alpha}} da$ [1].

Fractional conformal derivative concept has many applications in differential geometry, signal processing, mechanics and viscoelastic materials, electromagnetism, and optical systems [1, 5, 25, 36, 37, 49].

In Section 1, the introduction is presented. In Section 2, the intrinsic directional derivatives and generalized elastic lines along the $\mathfrak{Q}$ and $\mathcal{Z}$-lines directions of the Darboux frame in $\mathbb{R}^3$ are derived.

In Section 3, the congruence of surface $\Sigma^{(\mathfrak{Q})}$ containing the $\mathcal{T}-$and $\mathfrak{Q} - lines$ and the congruence of surface $\Sigma^{(\mathcal{Z})}$ containing the $\mathcal{T}-$ and $\mathcal{Z} - lines$ of the Darboux frame in $\mathbb{R}^3$ are presented. Recently, magnetic curves have been investigated by many researchers [2, 7, 9, 13, 44]. In this section, conformal magnetic curves, together with the conformal recursion, conformal normalization, and conformal differential formulas of maps associated with the conformal nonlinear Schrödinger equation (CNLSE) for the Darboux frame are derived. In the final section, conclusions are presented.

2 Intrinsic Derivatives and Generalized Elastic Line in the $\mathfrak{Q}$ and $\mathcal{Z}$-lines Directions of Darboux Frame in $\mathbb{R}^3$

The gradient operator ∇ is expressed:

$$\nabla = \mathbf{T}\frac{\delta}{\delta\mathfrak{T}} + \mathbf{Q}\frac{\delta}{\delta\mathfrak{Q}} + \mathbf{Z}\frac{\delta}{\delta\mathcal{Z}}.$$

$\frac{\delta}{\delta\mathfrak{T}}$, $\frac{\delta}{\delta\mathfrak{Q}}$ and $\frac{\delta}{\delta\mathcal{Z}}$ are the intrinsic directional derivatives in the $\mathbf{T}$, $\mathbf{Q}$ and $\mathbf{Z}$ directions, respectively for the Darboux frame in $\mathbb{R}^3$. The divergence and curl vectors of the frame $\{\mathbf{T}, \mathbf{Q}, \mathbf{Z}\}$ in $\mathbb{R}^3$ are

$$div\mathbf{T} = \left\langle \mathbf{Q}, \frac{\delta\mathbf{T}}{\delta\mathfrak{Q}} \right\rangle + \left\langle \mathbf{Z}, \frac{\delta\mathbf{T}}{\delta\mathcal{Z}} \right\rangle$$

$$= \psi^{(\mathfrak{TQ})} + \psi^{(\mathfrak{TZ})}$$

$$\begin{aligned} div\mathbf{Q} &= -\kappa_g + \left\langle \mathbf{Z}, \frac{\delta \mathbf{Q}}{\delta \mathcal{Z}} \right\rangle \\ div\mathbf{Z} &= -\kappa_n + \left\langle \mathbf{Q}, \frac{\delta \mathbf{Z}}{\delta \mathcal{Z}} \right\rangle \end{aligned}$$

$$curl\mathbf{T} = \xi^{(\mathfrak{T})}\mathbf{T} - \kappa_n\mathbf{Q} + \kappa_g\mathbf{Z}$$
$$curl\mathbf{Q} = -(div\mathbf{Z} + \kappa_n)\mathbf{T} + \xi^{(\mathfrak{Q})}\mathbf{Q} + \psi^{(\mathfrak{TQ})}\mathbf{Z}$$
$$curl\mathbf{Z} = (div\mathbf{Q} + \kappa_g)\mathbf{T} - \psi^{(\mathfrak{TZ})}\mathbf{Q} + \xi^{(\mathcal{Z})}\mathbf{Z}$$

where

$$\xi^{(\mathfrak{T})} = \langle curl\mathbf{T}, \mathbf{T} \rangle = \left\langle \mathbf{Z}, \frac{\delta \mathbf{T}}{\delta \mathfrak{Q}} \right\rangle - \left\langle \mathbf{Q}, \frac{\delta \mathbf{T}}{\delta \mathcal{Z}} \right\rangle$$

$$\xi^{(\mathfrak{Q})} = \langle curl\mathbf{Q}, \mathbf{Q} \rangle = -\tau_g + \left\langle \mathbf{T}, \frac{\delta \mathbf{Q}}{\delta \mathcal{Z}} \right\rangle$$

$$\xi^{(\mathcal{Z})} = \langle curl\mathbf{Z}, \mathbf{Z} \rangle = -\tau_g - \left\langle \mathbf{T}, \frac{\delta \mathbf{Z}}{\delta \mathfrak{Q}} \right\rangle$$

The intrinsic directional derivatives in the $\mathfrak{Q}$ and $\mathcal{Z}-$ lines directions for the Darboux frame in $\mathbb{R}^3$ are derived:

$$\frac{\delta}{\delta \mathfrak{Q}} \begin{bmatrix} \mathbf{T} \\ \mathbf{Q} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} 0 & \psi^{(\mathfrak{TQ})} & \tau_g + \xi^{(\mathcal{Z})} \\ -\psi^{(\mathfrak{TQ})} & 0 & -(div\mathbf{Z} + \kappa_n) \\ -(\tau_g + \xi^{(\mathcal{Z})}) & div\mathbf{Z} + \kappa_n & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{Q} \\ \mathbf{Z} \end{bmatrix} \tag{1}$$

$$\frac{\delta}{\delta \mathcal{Z}} \begin{bmatrix} \mathbf{T} \\ \mathbf{Q} \\ \mathbf{Z} \end{bmatrix} = \begin{bmatrix} 0 & -(\tau_g + \xi^{(\mathfrak{Q})}) & \psi^{(\mathfrak{TZ})} \\ \tau_g + \xi^{(\mathfrak{Q})} & 0 & div\mathbf{Q} + \kappa_g \\ -\psi^{(\mathfrak{TZ})} & -(div\mathbf{Q} + \kappa_g) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{Q} \\ \mathbf{Z} \end{bmatrix}. \tag{2}$$

2.1 The Congruence of the Surface $\Sigma^{(\mathfrak{Q})}$ Containing the $\mathfrak{T}$ and $\mathfrak{Q}$-lines

Consider

$$\frac{\delta \Sigma^{(\mathfrak{Q})}}{\delta \mathfrak{T}} = \mathbf{T}, \ \frac{\delta \Sigma^{(\mathfrak{Q})}}{\delta \mathfrak{Q}} = \mathbf{Q}.$$

The geodesic curvature in the direction $\mathfrak{Q} - lines$ is introduced:

$$\begin{aligned} {}^{(\mathfrak{Q})}\kappa_g &= \left\langle curl\mathbf{Q}, {}^{(\mathfrak{Q})}\mathbb{N} \right\rangle \\ &= \left\langle \frac{\delta^2 \Sigma^{(\mathfrak{Q})}}{\delta \mathfrak{Q}^2}, \mathbf{Z} \times \mathbf{Q} \right\rangle = \psi^{(\mathfrak{TQ})} \end{aligned}$$

Here, the unit normal vector field ${}^{(\mathfrak{Q})}\mathbb{N}$ of the surface $\Sigma^{(\mathfrak{Q})}$ is given by

$$^{(\mathfrak{Q})}\mathbb{N} = \frac{\frac{\delta\Sigma^{(\mathfrak{Q})}}{\delta\mathfrak{T}} \times \frac{\delta\Sigma^{(\mathfrak{Q})}}{\delta\mathfrak{Q}}}{\left\langle \frac{\delta\Sigma^{(\mathfrak{Q})}}{\delta\mathfrak{T}} \times \frac{\delta\Sigma^{(\mathfrak{Q})}}{\delta\mathfrak{Q}}, \frac{\delta\Sigma^{(\mathfrak{Q})}}{\delta\mathfrak{T}} \times \frac{\delta\Sigma^{(\mathfrak{Q})}}{\delta\mathfrak{Q}} \right\rangle^{1/2}} = \mathbf{Z}$$

The normal curvature for the congruence of the surface $\Sigma^{(\mathfrak{Q})}$, containing $\mathfrak{T}$ and $\mathfrak{Q} - lines$ in $\mathbb{R}^3$ is given by

$$^{(\mathfrak{Q})}\kappa_n = \left\langle \frac{\delta^2\Sigma^{(\mathfrak{Q})}}{\delta\mathfrak{Q}^2}, \mathbf{Z} \right\rangle = -(div\mathbf{Z} + \kappa_n).$$

The geodesic torsion for the congruence of surface $\Sigma^{(\mathfrak{Q})}$, containing $\mathfrak{T} - lines$ and $\mathfrak{Q} - lines$ is given by

$$^{(\mathfrak{Q})}\tau_g = \left\langle \frac{\delta\mathbf{Z}}{\delta\mathfrak{Q}}, \mathbf{Z} \times \mathbf{Q} \right\rangle = \tau_g + \xi^{(\mathcal{Z})}$$

2.2 The Congruence of the Surface $\Sigma^{(\mathcal{Z})}$ containing the $\mathfrak{T}$ and $\mathcal{Z}$-lines

Consider

$$\frac{\delta\Sigma^{(\mathcal{Z})}}{\delta\mathfrak{T}} = \mathbf{T}, \ \frac{\delta\Sigma^{(\mathcal{Z})}}{\delta\mathcal{Z}} = \mathbf{Z}$$

The geodesic curvature of the surface $\Sigma^{(\mathcal{Z})}$ is given by

$$^{(\mathcal{Z})}\kappa_g = \left\langle curl\mathbf{Z}, ^{(\mathcal{Z})}\mathbb{N} \right\rangle = \psi^{(\mathfrak{T}\mathcal{Z})}$$

Here, the unit normal vector field of the surface $\Sigma^{(\mathcal{Z})}$ is obtained:

$$^{(\mathcal{Z})}\mathbb{N} = \frac{\frac{\delta\Sigma^{(\mathcal{Z})}}{\delta\mathfrak{T}} \times \frac{\delta\Sigma^{(\mathcal{Z})}}{\delta\mathcal{Z}}}{\left\langle \frac{\delta\Sigma^{(\mathcal{Z})}}{\delta\mathfrak{T}} \times \frac{\delta\Sigma^{(\mathcal{Z})}}{\delta\mathcal{Z}}, \frac{\delta\Sigma^{(\mathcal{Z})}}{\delta\mathfrak{T}} \times \frac{\delta\Sigma^{(\mathcal{Z})}}{\delta\mathcal{Z}} \right\rangle^{1/2}} = -\mathbf{Q}$$

The normal curvature of the surface $\Sigma^{(\mathcal{Z})}$ is given by

$$^{(\mathcal{Z})}\kappa_n = \left\langle \frac{\delta^2\Sigma^{(\mathcal{Z})}}{\delta\mathcal{Z}^2}, -\mathbf{Q} \right\rangle = \left\langle \frac{\delta\mathbf{Z}}{\delta\mathcal{Z}}, -\mathbf{Q} \right\rangle$$

The geodesic torsion of the surface $\Sigma^{(\mathcal{Z})}$ is given by

$$^{(\mathcal{Z})}\tau_g = \left\langle \frac{\delta\mathbf{Q}}{\delta\mathcal{Z}}, \mathbf{Q} \times \mathbf{Z} \right\rangle = \tau_g + \xi^{(\mathfrak{Q})}$$

2.3 Generalized Elastic Line and Obtaining of the Intrinsic Directional Derivatives in the Direction of $\mathfrak{Q}$-lines for the Darboux Frame in $\mathbb{R}^3$

The arc γ is called generalized elastic line if it is extremal for the minimizing the functional

$$\mathcal{I}_1 = \int_0^l \kappa^2\tau^{\mathfrak{Q}}\delta\mathfrak{Q} \tag{3}$$

in the family of all arcs of the length l on the surface having the same initial point and initial direction as γ in the direction of $\mathfrak{Q}-lines$ for Darboux frame in $\mathbb{R}^3$. The curvature and torsion in the $\mathfrak{Q}-lines$ direction for the Darboux frame in $\mathbb{R}^3$ are

$$\mathfrak{Q}_K = \frac{\left\langle \frac{\delta\gamma}{\delta\mathfrak{Q}} \times \frac{\delta^2\gamma}{\delta\mathfrak{Q}^2}, \frac{\delta\gamma}{\delta\mathfrak{Q}} \times \frac{\delta^2\gamma}{\delta\mathfrak{Q}^2} \right\rangle^{1/2}}{\left\langle \frac{\delta\gamma}{\delta\mathfrak{Q}}, \frac{\delta\gamma}{\delta\mathfrak{Q}} \right\rangle^{3/2}}, \quad \tau^{\mathfrak{Q}} = \frac{\left\langle \frac{\delta\gamma}{\delta\mathfrak{Q}} \times \frac{\delta^2\gamma}{\delta\mathfrak{Q}^2}, \frac{\delta^3\gamma}{\delta\mathfrak{Q}^3} \right\rangle}{\left\langle \frac{\delta\gamma}{\delta\mathfrak{Q}} \times \frac{\delta^2\gamma}{\delta\mathfrak{Q}^2}, \frac{\delta\gamma}{\delta\mathfrak{Q}} \times \frac{\delta^2\gamma}{\delta\mathfrak{Q}^2} \right\rangle}$$

Consider the arc γ determined by

$$\gamma(\mathfrak{Q}) = x(u(\mathfrak{Q}), v(\mathfrak{Q})),\ 0 \leq \mathfrak{Q} \leq l$$

Moreover,

$$\mathbf{Q}(\mathfrak{Q}) \quad = \quad \eta_1(\mathfrak{Q})\frac{\delta x}{\delta u} + \eta_2(\mathfrak{Q})\frac{\delta x}{\delta v}$$

$\eta_1(\mathfrak{Q})$ and $\eta_2(\mathfrak{Q})$ are suitable scalar functions in the direction of $\mathfrak{Q}$-lines for Darboux frame in $\mathbb{R}^3$. The variational arcs of length l are obtained with extend the arc γ to the arc γ^* for $0 \leq \mathfrak{Q} \leq l^*, l^* > l$. Define

$$\eta(\mathfrak{Q}) = \mu\eta_1(\mathfrak{Q}),\ \varpi^*(\mathfrak{Q}) = \mu\eta_2(\mathfrak{Q})$$

$$\eta(\mathfrak{Q})\frac{\delta x}{\delta u} + \varpi^*(\mathfrak{Q})\frac{\delta x}{\delta v} = \mu(\mathfrak{Q})\mathbf{Q}(\mathfrak{Q})$$

along γ. Suppose that

$$\mu(0) = 0,\ \frac{\delta\mu}{\delta\mathfrak{Q}}(0) = 0,\ \frac{\delta^2\mu}{\delta\mathfrak{Q}^2}(0) = 0 \tag{4}$$

Define

$$\varphi(\mathfrak{Q}^*; u) = x\left(u(\mathfrak{Q}) + u\eta(\mathfrak{Q}^*), v(\mathfrak{Q}^*) + u\varpi^*(\mathfrak{Q}^*)\right)$$

for $0 \leq \mathfrak{Q}^* \leq l^*$. For $|u| < \epsilon$, $\varphi(\mathfrak{Q}^*; u)$ lies in the coordinate patch in the $\mathfrak{Q}-lines$ direction. $\varphi(\mathfrak{Q}^*; 0)$ is the same with γ^* and $\mathfrak{Q}^*$ is the arc length with (4) in the direction of $\mathfrak{Q}-lines$ for the Darboux frame in $\mathbb{R}^3$. $\mathbb{L}^*(u)$ denotes the length of the arc $\varphi(\mathfrak{Q}^*; u), 0 \leq \mathfrak{Q}^* \leq l^*$. Then

$$\mathbb{L}^*(u) = \int_0^{l^*} \left\langle \frac{\delta\varphi}{\delta\mathfrak{Q}^*}(\mathfrak{Q}^*; u), \frac{\delta\varphi}{\delta\mathfrak{Q}^*}(\mathfrak{Q}^*; u) \right\rangle^{1/2} \delta\mathfrak{Q}^*$$

and

$$\mathbb{L}^*(0) = l^* > l,\ \mathbb{L}^*(u) > \frac{l + l^*}{2} > l$$

for a suitable ϵ_1 satisfying $0 < \epsilon_1 \leq \epsilon$ and $0 \leq \mathfrak{Q}^* \leq \lambda(u) \leq l^*$, $0 \leq |u| \leq \epsilon_1$. Furthermore,

$$\int_0^{\lambda(u)} \left\langle \frac{\delta\varphi}{\delta\mathfrak{Q}^*}, \frac{\delta\varphi}{\delta\mathfrak{Q}^*} \right\rangle^{1/2} \delta\mathfrak{Q}^* = l.$$

$\lambda(0) = l$. Additionally,

$$\left.\frac{\delta\lambda}{\delta u}\right|_{u=0} = \int_0^l \mu\psi^{(\mathfrak{T}\mathfrak{Q})}\,\delta\mathfrak{Q}$$

$$\left.\frac{\delta\varphi}{\delta\mathfrak{Q}^*}\right|_{u=0} = \mathbf{T}, \qquad 0 \leq \mathfrak{Q}^* \leq l$$

$$\left.\frac{\delta^2\varphi}{\delta\mathfrak{Q}^{*2}}\right|_{u=0} = \psi^{(\mathfrak{T}\mathfrak{Q})}\mathbf{Q} - (\kappa_n + div\mathbf{Z})\mathbf{Z}.$$

$$\begin{aligned}\left.\frac{\delta^3\varphi}{\delta\mathfrak{Q}^{*3}}\right|_{u=0} &= -((\psi^{(\mathfrak{T}\mathfrak{Q})})^2 + (\kappa_n + div\mathbf{Z})^2)\mathbf{T} + (\frac{\delta(\psi^{(\mathfrak{T}\mathfrak{Q})})}{\delta\mathfrak{Q}} \\ &\quad +(\kappa_n + div\mathbf{Z})(\tau_g + \xi^{(\mathcal{Z})}))\mathbf{Q} \\ &\quad +(\psi^{(\mathfrak{T}\mathfrak{Q})}(\tau_g + \xi^{(\mathcal{Z})}) - \frac{\delta(\kappa_n + div\mathbf{Z})}{\delta\mathfrak{Q}})\mathbf{Z}\end{aligned}$$

$$\left.\frac{\delta\varphi}{\delta u}\right|_{u=0} = \mu\mathbf{Q}.$$

$$\left.\frac{\delta^2\varphi}{\delta u\delta\mathfrak{Q}^*}\right|_{u=0} = -\mu\psi^{(\mathfrak{T}\mathfrak{Q})}\mathbf{T} + \frac{\delta\mu}{\delta\mathfrak{Q}}\mathbf{Q} + \mu(\tau_g + \xi^{(\mathcal{Z})})\mathbf{Z}$$

$$\begin{aligned}\left.\frac{\delta^3\varphi}{\delta u\delta\mathfrak{Q}^{*2}}\right|_{u=0} &= \left(-2\frac{\delta\mu}{\delta\mathfrak{Q}}\psi^{(\mathfrak{T}\mathfrak{Q})} - \mu\frac{\delta(\psi^{(\mathfrak{T}\mathfrak{Q})})}{\delta\mathfrak{Q}} + \mu(\tau_g + \xi^{(\mathcal{Z})})(\kappa_n + div\mathbf{Z})\right)\mathbf{T} \\ &\quad + \left(\frac{\delta^2\mu}{\delta\mathfrak{Q}^2} - \mu(\psi^{(\mathfrak{T}\mathfrak{Q})})^2 - \mu(\kappa_n + div\mathbf{Z})^2\right)\mathbf{Q} \\ &\quad + \left(2\frac{\delta\mu}{\delta\mathfrak{Q}}(\tau_g + \xi^{(\mathcal{Z})}) + \mu\psi^{(\mathfrak{T}\mathfrak{Q})}(\kappa_n + div\mathbf{Z}) + \mu\frac{\delta(\tau_g + \xi^{(\mathcal{Z})})}{\delta\mathfrak{Q}}\right)\mathbf{Z}.\end{aligned}$$

$$\left.\frac{\delta^4\varphi}{\delta u\delta\mathfrak{Q}^{*3}}\right|_{u=0} = \begin{pmatrix} \mu(\psi^{(\mathfrak{T}\mathfrak{Q})})^3 - 3\frac{\delta\mu}{\delta\mathfrak{Q}}\frac{\delta(\psi^{(\mathfrak{T}\mathfrak{Q})})}{\delta\mathfrak{Q}} - 3\frac{\delta^2\mu}{\delta\mathfrak{Q}^2}\psi^{(\mathfrak{T}\mathfrak{Q})} \\ +3\frac{\delta\mu}{\delta\mathfrak{Q}}(\tau_g + \xi^{(\mathcal{Z})})(\kappa_n + div\mathbf{Z}) + 2\mu\frac{\delta(\tau_g + \xi^{(\mathcal{Z})})}{\delta\mathfrak{Q}}(\kappa_n + div\mathbf{Z}) \\ -\mu(\tau_g + \xi^{(\mathcal{Z})})\frac{\delta(\kappa_n + div\mathcal{Z})}{\delta\mathfrak{Q}} + \mu(\tau_g + \xi^{(\mathcal{Z})})^2\psi^{(\mathfrak{T}\mathfrak{Q})} \\ +\mu\psi^{(\mathfrak{T}\mathfrak{Q})}(\kappa_n + div\mathbf{Z})^2 - \mu\frac{\delta^2(\psi^{(\mathfrak{T}\mathfrak{Q})})}{\delta\mathfrak{Q}^2} \end{pmatrix}\mathbf{T}$$

$$+\begin{pmatrix} \frac{\delta^3\mu}{\delta\mathfrak{Q}^3} - 3\frac{\delta\mu}{\delta\mathfrak{Q}}(\psi^{(\mathfrak{T}\mathfrak{Q})})^2 - 3\frac{\delta\mu}{\delta\mathfrak{Q}}(\tau_g+\xi^{(\mathcal{Z})})^2 \\ -3\mu\psi^{(\mathfrak{T}\mathfrak{Q})}\frac{\delta(\psi^{(\mathfrak{T}\mathfrak{Q})})}{\delta\mathfrak{Q}} - 3\mu(\tau_g+\xi^{(\mathcal{Z})})\frac{\delta(\tau_g+\xi^{(\mathcal{Z})})}{\delta\mathfrak{Q}} \end{pmatrix}\mathbf{Q}$$

$$+\begin{pmatrix} 3\frac{\delta^2\mu}{\delta\mathfrak{Q}^2}(\tau_g+\xi^{(\mathcal{Z})}) + 3\frac{\delta\mu}{\delta\mathfrak{Q}}\frac{\delta(\tau_g+\xi^{(\mathcal{Z})})}{\delta\mathfrak{Q}} \\ +3\frac{\delta\mu}{\delta\mathfrak{Q}}\psi^{(\mathfrak{T}\mathfrak{Q})}(\kappa_n + div\mathbf{Z}) \\ \mu(-\psi^{(\mathfrak{T}\mathfrak{Q})}\frac{\delta(\kappa_n+div\mathbf{Z})}{\delta\mathfrak{Q}} + 2\frac{\delta(\psi^{(\mathfrak{T}\mathfrak{Q})})}{\delta\mathfrak{Q}}(\kappa_n+div\mathbf{Z}) \\ -(\tau_g+\xi^{(\mathcal{Z})})^3 - (\psi^{(\mathfrak{T}\mathfrak{Q})})^2(\tau_g+\xi^{(\mathcal{Z})}) \\ +\frac{\delta^2(\tau_g+\xi^{(\mathcal{Z})})}{\delta\mathfrak{Q}^2} - (\tau_g+\xi^{(\mathcal{Z})})(\kappa_n+div\mathbf{Z})^2 \end{pmatrix}\mathbf{Z}$$

In the $\mathfrak{Q}-lines$ direction along γ, it is obtained

$$\left.\frac{\delta\lambda}{\delta u}\right|_{u=0} = \int_0^l \mu\psi^{(\mathfrak{T}\mathfrak{Q})}\delta\mathfrak{Q} \tag{5}$$

From (5),

$$\begin{aligned} &\left.\frac{\delta\lambda}{\delta u}\right|_{u=0}\sqrt{\left\langle \left.\frac{\delta\varphi}{\delta\mathfrak{Q}^*}\right|_{u=0}, \left.\frac{\delta\varphi}{\delta\mathfrak{Q}^*}\right|_{u=0}\right\rangle} \\ &+\int_0^l \left\langle \left.\frac{\delta\varphi}{\delta\mathfrak{Q}^*}\right|_{u=0}, \left.\frac{\delta^2\varphi}{\delta\mathfrak{Q}^*\delta u}\right|_{u=0}\right\rangle \left\langle \left.\frac{\delta\varphi}{\delta\mathfrak{Q}^*}\right|_{u=0}, \left.\frac{\delta\varphi}{\delta\mathfrak{Q}^*}\right|_{u=0}\right\rangle^{-1/2}\delta\mathfrak{Q}^* = 0 \end{aligned} \tag{6}$$

In the $\mathfrak{Q}-lines$ direction,

$$\mathcal{I}_1(u) = \int_0^{\lambda(u)} \left\langle \frac{\delta\varphi}{\delta\mathfrak{Q}^*}\wedge\frac{\delta^2\varphi}{\delta\mathfrak{Q}^{*2}}, \frac{\delta^3\varphi}{\delta\mathfrak{Q}^{*3}}\right\rangle \left\langle \frac{\delta\varphi}{\delta\mathfrak{Q}^*}, \frac{\delta\varphi}{\delta\mathfrak{Q}^*}\right\rangle^{-5/2}\delta\mathfrak{Q}^*. \tag{7}$$

From (7)

$$\begin{aligned} \frac{\delta\mathcal{I}_1}{\delta u}(u) &= \frac{\delta\lambda}{\delta u}\left\{\left\langle \frac{\delta\varphi}{\delta\mathfrak{Q}^*}\wedge\frac{\delta^2\varphi}{\delta\mathfrak{Q}^{*2}}, \frac{\delta^3\varphi}{\delta\mathfrak{Q}^{*3}}\right\rangle \left\langle \frac{\delta\varphi}{\delta\mathfrak{Q}^*}, \frac{\delta\varphi}{\delta\mathfrak{Q}^*}\right\rangle^{-5/2}\right\}_{\mathfrak{Q}^*=\lambda(u)} \\ &+\int_0^{\lambda(u)}\left\{\begin{array}{c}\left(\left\langle \frac{\delta^2\varphi}{\delta u\delta\mathfrak{Q}^*}\wedge\frac{\delta^2\varphi}{\delta\mathfrak{Q}^{*2}}, \frac{\delta^3\varphi}{\delta\mathfrak{Q}^{*3}}\right\rangle + \left\langle \frac{\delta\varphi}{\delta\mathfrak{Q}^*}\wedge\frac{\delta^3\varphi}{\delta u\delta\mathfrak{Q}^{*2}}, \frac{\delta^3\varphi}{\delta\mathfrak{Q}^{*3}}\right\rangle\right. \\ \left.+\left\langle \frac{\delta\varphi}{\delta\mathfrak{Q}^*}\wedge\frac{\delta^2\varphi}{\delta\mathfrak{Q}^{*2}}, \frac{\delta^4\varphi}{\delta u\delta\mathfrak{Q}^{*3}}\right\rangle\right)\left\langle \frac{\delta\varphi}{\delta\mathfrak{Q}^*}, \frac{\delta\varphi}{\delta\mathfrak{Q}^*}\right\rangle^{-5/2}\end{array}\right\}\delta\mathfrak{Q}^* \\ &-5\int_0^{\lambda(u)}\left\langle \frac{\delta\varphi}{\delta\mathfrak{Q}^*}\wedge\frac{\delta^2\varphi}{\delta\mathfrak{Q}^{*2}}, \frac{\delta^3\varphi}{\delta\mathfrak{Q}^{*3}}\right\rangle \left\langle \frac{\delta\varphi}{\delta\mathfrak{Q}^*}, \frac{\delta\varphi}{\delta\mathfrak{Q}^*}\right\rangle^{-7/2}\left\langle \frac{\delta^2\varphi}{\delta u\delta\mathfrak{Q}^*}, \frac{\delta\varphi}{\delta\mathfrak{Q}^*}\right\rangle\delta\mathfrak{Q}^* \end{aligned}$$

For $u = 0$,

$$\frac{\delta \mathcal{I}_1}{\delta u}(0) = \int_0^l \mu \left\{ \begin{array}{c} -2\frac{\delta^3(\kappa_n+div\mathbf{Z})}{\delta\mathfrak{Q}^3} + 6(\tau_g+\xi^{(\mathcal{Z})})\frac{\delta^2(\psi^{(\mathfrak{T}\mathfrak{Q})})}{\delta\mathfrak{Q}^2} + (\psi^{(\mathfrak{T}\mathfrak{Q})})^2(l)(\tau_g+\xi^{(\mathcal{Z})})(l) \\ -3(\kappa_n+div\mathbf{Z})^2\frac{\delta(\kappa_n+cdiv\mathcal{Z})}{\delta\mathfrak{Q}} + 3(\psi^{(\mathfrak{T}\mathfrak{Q})})^2\frac{\delta(\kappa_n+div\mathbf{Z})}{\delta\mathfrak{Q}} \\ -2(\psi^{(\mathfrak{T}\mathfrak{Q})})(\tau_g+\xi^{(\mathcal{Z})})^3 + 2(\psi^{(\mathfrak{T}\mathfrak{Q})})\frac{\delta^2(\tau_g+\xi^{(\mathcal{Z})})}{\delta\mathfrak{Q}^2} \\ +6(\kappa_n+div\mathbf{Z})(\tau_g+\xi^{(\mathcal{Z})})\frac{\delta(\tau_g+\xi^{(\mathcal{Z})})}{\delta\mathfrak{Q}} - 6(\tau_g+\xi^{(\mathcal{Z})})^2\frac{\delta(\kappa_n+div\mathbf{Z})}{\delta\mathfrak{Q}} \\ +3(\psi^{(\mathfrak{T}\mathfrak{Q})})(\tau_g+\xi^{(\mathcal{Z})})(\kappa_n+div\mathbf{Z})^2 + 3(\psi^{(\mathfrak{T}\mathfrak{Q})})^3(\tau_g+\xi^{(\mathcal{Z})}) \\ +6\frac{\delta(\psi^{(\mathfrak{T}\mathfrak{Q})})}{\delta\mathfrak{Q}}\frac{\delta(\tau_g+\xi^{(\mathcal{Z})})}{\delta\mathfrak{Q}} + (\psi^{(\mathfrak{T}\mathfrak{Q})})((\kappa_n+div\mathbf{Z})^2(l))(\tau_g+\xi^{(\mathcal{Z})})(l) \\ -(\kappa_n+div\mathbf{Z})(l)\frac{\delta(\psi^{(\mathfrak{T}\mathfrak{Q})})}{\delta\mathfrak{Q}} - (\psi^{(\mathfrak{T}\mathfrak{Q})})(l)\frac{\delta(\kappa_n+div\mathbf{Z})}{\delta\mathfrak{Q}}(l)) \end{array} \right\} \delta\mathfrak{Q}$$

$$+\mu(l)\left(\begin{array}{c} -6(\tau_g+\xi^{(\mathcal{Z})})(l)\frac{\delta(\psi^{(\mathfrak{T}\mathfrak{Q})})}{\delta\mathfrak{Q}}(l) - (\psi^{(\mathfrak{T}\mathfrak{Q})})(l)\frac{\delta(\tau_g+\xi^{(\mathcal{Z})})}{\delta\mathfrak{Q}}(l) \\ -2\frac{\delta^2(\kappa_n+div\mathcal{Z})}{\delta\mathfrak{Q}^2} - 5(\kappa_n+div\mathcal{Z})(l)(\tau_g+\xi^{(\mathcal{Z})})^2(l) \\ -(\psi^{(\mathfrak{T}\mathfrak{Q})})^2(l)(\kappa_n+div\mathcal{Z})(l) + (\kappa_n+div\mathbf{Z})^3(l) \end{array} \right)$$

$$+\frac{\delta\mu}{\delta\mathfrak{Q}}\left(4(\psi^{(\mathfrak{T}\mathfrak{Q})})(l)(\tau_g+\xi^{(\mathcal{Z})})(l) - 2\frac{\delta(\kappa_n+div\mathbf{Z})}{\delta\mathfrak{Q}}(l)\right)$$

$$-\frac{\delta^2\mu}{\delta\mathfrak{Q}^2}(l)(\kappa_n+div\mathbf{Z})(l)$$

$\frac{\delta \mathcal{I}_1}{\delta u}(0) = 0$ for all choices of the function μ with arbitrary values of $\mu(l)$, $\frac{\delta\mu}{\delta\mathfrak{Q}}(l)$ and $\frac{\delta^2\mu}{\delta\mathfrak{Q}^2}(l)$ and the path of the arc γ must satisfy as the following boundary conditions ${}^{(\mathfrak{Q})}(\mathfrak{BC}1)$, ${}^{(\mathfrak{Q})}(\mathfrak{BC}2)$, ${}^{(\mathfrak{Q})}(\mathfrak{BC}3)$ and differential equation ${}^{(\mathfrak{Q})}(\mathfrak{DE})$ in the $\mathfrak{Q}-lines$ direction.

$${}^{(\mathfrak{Q})}(\mathfrak{BC}1) \quad (\kappa_n+div\mathbf{Z})(l) = 0$$

$${}^{(\mathfrak{Q})}(\mathfrak{BC}2) \quad \frac{\delta(\kappa_n+div\mathbf{Z})}{\delta\mathfrak{Q}}(l) = 2(\psi^{(\mathfrak{T}\mathfrak{Q})})(l)(\tau_g+\xi^{(\mathcal{Z})})(l)$$

$${}^{(\mathfrak{Q})}(\mathfrak{BC}3) \quad \left\{ \begin{array}{l} 2\frac{\delta^2(\kappa_n+div\mathbf{Z})}{\delta\mathfrak{Q}^2} = -6(\tau_g+\xi^{(\mathcal{Z})})(l)\frac{\delta(\psi^{(\mathfrak{T}\mathfrak{Q})})}{\delta\mathfrak{Q}}(l) - (\psi^{(\mathfrak{T}\mathfrak{Q})})(l)\frac{\delta(\tau_g+\xi^{(\mathcal{Z})})}{\delta\mathfrak{Q}}(l) \\ -5(\kappa_n+div\mathbf{Z})(l)(\tau_g+\xi^{(\mathcal{Z})})^2(l) - (\psi^{(\mathfrak{T}\mathfrak{Q})})^2(l)(\kappa_n+div\mathbf{Z})(l) \\ +(\kappa_n+div\mathbf{Z})^3(l) \end{array} \right.$$

$$
{}^{(\mathfrak{Q})}(\mathfrak{D}\mathfrak{E}) \quad \left\{ \begin{array}{c}
-2\frac{\delta^3(\kappa_n+div\mathbf{Z})}{\delta\mathfrak{Q}^3}+6(\tau_g+\xi^{(\mathcal{Z})})\frac{\delta^2(\psi^{(\mathfrak{TQ})})}{\delta\mathfrak{Q}^2}+(\psi^{(\mathfrak{TQ})})^2(l)(\tau_g+\xi^{(\mathcal{Z})})(l) \\
-3(\kappa_n+div\mathbf{Z})^2\frac{\delta(\kappa_n+div\mathcal{Z})}{\delta\mathfrak{Q}}+3(\psi^{(\mathfrak{TQ})})^2\frac{\delta(\kappa_n+div\mathbf{Z})}{\delta\mathfrak{Q}} \\
-2(\psi^{(\mathfrak{TQ})})(\tau_g+\xi^{(\mathcal{Z})})^3+2(\psi^{(\mathfrak{TQ})})\frac{\delta^2(\tau_g+\xi^{(\mathcal{Z})})}{\delta\mathfrak{Q}^2} \\
+6(\kappa_n+div\mathbf{Z})(\tau_g+\xi^{(\mathcal{Z})})\frac{\delta(\tau_g+\xi^{(\mathcal{Z})})}{\delta\mathfrak{Q}}-6(\tau_g+\xi^{(\mathcal{Z})})^2\frac{\delta(\kappa_n+div\mathbf{Z})}{\delta\mathfrak{Q}} \\
+3(\psi^{(\mathfrak{TQ})})(\tau_g+\xi^{(\mathcal{Z})})(\kappa_n+div\mathcal{Z})^2+3(\psi^{(\mathfrak{TQ})})^3(\tau_g+\xi^{(\mathcal{Z})}) \\
+6\frac{\delta(\psi^{(\mathfrak{TQ})})}{\delta\mathfrak{Q}}\frac{\delta(\tau_g+\xi^{(\mathcal{Z})})}{\delta\mathfrak{Q}}+(\psi^{(\mathfrak{TQ})})((\kappa_n+div\mathbf{Z})^2(l))(\tau_g+\xi^{(\mathcal{Z})})(l) \\
-(\kappa_n+div\mathbf{Z})(l)\frac{\delta(\psi^{(\mathfrak{TQ})})}{\delta\mathfrak{Q}}-(\psi^{(\mathfrak{TQ})})(l)\frac{\delta(\kappa_n+div\mathbf{Z})}{\delta\mathfrak{Q}}(l))=0
\end{array} \right.
$$

2.4 Generalized Elastic Line and Obtaining of Intrinsic Derivatives in the Direction of $\mathcal{Z}$-lines for Darboux Frame in $\mathbb{R}^3$

The arc γ is called generalized elastic line if it is extremal for the minimizing the following functional

$$\mathcal{I}_2=\int_0^l({}^{\mathcal{Z}}\kappa)^2\tau^{\mathcal{Z}}\delta\mathcal{Z} \tag{8}$$

in the family of all arcs of l on the surface having the same initial point and initial direction as γ, where γ lies in the direction of $\mathcal{Z}-lines$ for the Darboux frame in $\mathbb{R}^3$. The curvature and torsion for the curve γ in the $\mathcal{Z}-lines$ direction are

$${}^{\mathcal{Z}}\kappa=\frac{\left\langle\frac{\delta\gamma}{\delta\mathcal{Z}}\times\frac{\delta^2\gamma}{\delta\mathcal{Z}^2},\frac{\delta\gamma}{\delta\mathcal{Z}}\times\frac{\delta^2\gamma}{\delta\mathcal{Z}^2}\right\rangle^{1/2}}{\left\langle\frac{\delta\gamma}{\delta\mathcal{Z}},\frac{\delta\gamma}{\delta\mathcal{Z}}\right\rangle^{3/2}},\quad \tau^{\mathcal{Z}}=\frac{\left\langle\frac{\delta\gamma}{\delta\mathcal{Z}}\times\frac{\delta^2\gamma}{\delta\mathcal{Z}^2},\frac{\delta^3\gamma}{\delta\mathcal{Z}^3}\right\rangle}{\left\langle\frac{\delta\gamma}{\delta\mathcal{Z}}\times\frac{\delta^2\gamma}{\delta\mathcal{Z}^2},\frac{\delta\gamma}{\delta\mathcal{Z}}\times\frac{\delta^2\gamma}{\delta\mathcal{Z}^2}\right\rangle}$$

The arc γ is determined by

$$\gamma(\mathcal{Z})=x(u(\mathcal{Z}),v(\mathcal{Z})),\ 0\le\mathcal{Z}\le l$$

Moreover,

$$\mathbf{Q}(\mathcal{Z}) \ = \ \eta_1(\mathcal{Z})\frac{\delta x}{\delta u}+\eta_2(\mathcal{Z})\frac{\delta x}{\delta v}$$

Here, $\eta_1(\mathcal{Z})$ and $\eta_2(\mathcal{Z})$ are suitable scalar functions in the direction of $\mathcal{Z}-lines$ in $\mathbb{R}^3$. The variational arcs of length l are obtained with extend the arc γ to the arc γ^* for $0\le\mathcal{Z}\le l^*$, $l^*>l$. Define

$$\eta(\mathcal{Z})=\mu(\mathcal{Z})\eta_1(\mathcal{Z}),\ \varpi^*(\mathcal{Z})=\mu(\mathcal{Z})\eta_2(\mathcal{Z})$$

$$\eta(\mathcal{Z})\frac{\delta x}{\delta u} + \varpi^*(\mathcal{Z})\frac{\delta x}{\delta v} = \mu(\mathcal{Z})\mathbf{Q}(\mathcal{Z})$$

along γ. Suppose that

$$\mu(0) = 0, \frac{\delta\mu}{\delta\mathcal{Z}}(0) = 0, \frac{\delta^2\mu}{\delta\mathcal{Z}^2}(0) = 0 \tag{9}$$

Define

$$\varphi(\mathcal{Z}^*; u) = x\left(u(\mathcal{Z}^*) + u\eta(\mathcal{Z}^*), v(\mathcal{Z}^*) + u\varpi^*(\mathcal{Z}^*)\right)$$

for $0 \le \mathcal{Z}^* \le l^*$. For $|u| < \epsilon$, $\varphi(\mathcal{Z}^*; u)$ lies in the coordinate patch in the direction of $\mathcal{Z}-lines.$ $\varphi(\mathcal{Z}^*; 0)$ is the same as the arc γ^* and $\mathcal{Z}^*$ is the arc length in the direction $\mathcal{Z}-lines.$ $\mathbb{L}^*(u)$ denotes the length of the arc $\varphi(\mathcal{Z}^*; u), 0 \le \mathcal{Z}^* \le l^*$. Then

$$\mathbb{L}^*(u) = \int_0^{l^*} \left\langle \frac{\delta\varphi}{\delta\mathcal{Z}^*}(\mathcal{Z}^*; u), \frac{\delta\varphi}{\delta\mathcal{Z}^*}(\mathcal{Z}^*; u) \right\rangle^{1/2} \delta\mathcal{Z}^* \tag{10}$$

Additionally,

$$\mathbb{L}^*(0) = l^* > l,\ \mathbb{L}^*(u) > \frac{l + l^*}{2} > l,\ \ |u| < \epsilon_1$$

for a suitable ϵ_1 satisfying $0 < \epsilon_1 \le \epsilon$. Moreover, $0 \le \mathcal{Z}^* \le \lambda(u) \le l^*, 0 \le |u| \le \epsilon_1$

$$\int_0^{\lambda(u)} \left\langle \frac{\delta\varphi}{\delta\mathcal{Z}^*}, \frac{\delta\varphi}{\delta\mathcal{Z}^*} \right\rangle^{1/2} \delta\mathcal{Z}^* = l.$$

$\lambda(0) = l$. Also,

$$\left.\frac{\delta\lambda}{\delta u}\right|_{u=0} = \int_0^l \mu\psi^{(\mathfrak{T}\mathcal{Z})}\delta\mathcal{Z} \tag{11}$$

$$\left.\frac{\delta\varphi}{\delta\mathcal{Z}^*}\right|_{u=0} = \mathbf{T} \qquad 0 \le \mathcal{Z}^* \le l$$

$$\left.\frac{\delta^2\varphi}{\delta\mathcal{Z}^{*2}}\right|_{u=0} = \psi^{(\mathfrak{T}\mathcal{Z})}\mathbf{Q} - (\kappa_g + div\mathbf{Q})\mathbf{Z}.$$

$$\begin{aligned}\left.\frac{\delta^3\varphi}{\delta\mathcal{Z}^{*3}}\right|_{u=0} = & -\left((\psi^{(\mathfrak{T}\mathcal{Z})})^2 + (\operatorname{div}\mathbf{Q} + \kappa_g)^2\right)\mathbf{T} \\ & + \left(\frac{\delta\psi^{(\mathfrak{T}\mathcal{Z})}}{\delta\mathcal{Z}} + (\kappa_g + \operatorname{div}\mathbf{Q})(\tau_g + \xi^{(\mathfrak{Q})})\right)\mathbf{Q} \\ & + \left(\psi^{(\mathfrak{T}\mathcal{Z})}(\tau_g + \xi^{(\mathfrak{Q})}) - \frac{\delta(\kappa_g + \operatorname{div}\mathbf{Q})}{\delta\mathcal{Z}}\right)\mathbf{Z}\end{aligned}$$

Furthermore,

$$\left.\frac{\delta\varphi}{\delta u}\right|_{u=0} = \mu\mathbf{Q}.$$

$$\left.\frac{\delta^2\varphi}{\delta u\delta\mathcal{Z}^*}\right|_{u=0} = -\mu\psi^{(\mathfrak{T}\mathcal{Z})}\mathbf{T} + \frac{\delta\mu}{\delta\mathcal{Z}}\mathbf{Q} + \mu(\tau_g + \xi^{(\mathfrak{Q})})\mathbf{Z}$$

$$\begin{aligned}\left.\frac{\delta^3\varphi}{\delta u\delta\mathcal{Z}^{*2}}\right|_{u=0} &= \left(-2\frac{\delta\mu}{\delta\mathcal{Z}}\psi^{(\mathfrak{T}\mathcal{Z})} - \mu\frac{\delta\psi^{(\mathfrak{T}\mathcal{Z})}}{\delta\mathcal{Z}} + \mu(\tau_g + \psi^{(\mathfrak{T}\mathfrak{Q})})(\kappa_g + div\mathbf{Q})\right)\mathbf{T}\\ &+ \left(\frac{\delta^2\mu}{\delta\mathcal{Z}^2} - \mu(\psi^{(\mathfrak{T}\mathcal{Z})})^2 - \mu(\tau_g + \xi^{(\mathfrak{Q})})^2\right)\mathbf{Q}\\ &+ \left(2\frac{\delta\mu}{\delta\mathcal{Z}}(\tau_g + \xi^{(\mathfrak{Q})}) + \mu\psi^{(\mathfrak{T}\mathcal{Z})}(\kappa_g + div\mathbf{Q}) + \mu\frac{\delta(\tau_g + \xi^{(\mathfrak{Q})})}{\delta\mathcal{Z}}\right)\mathbf{Z}\end{aligned}$$

$$\begin{aligned}\left.\frac{\delta^4\varphi}{\delta u\delta\mathcal{Z}^{*3}}\right|_{u=0} &= \begin{pmatrix} \mu(\psi^{(\mathfrak{T}\mathcal{Z})})^3 - 3\frac{\delta\mu}{\delta\mathcal{Z}}\frac{\delta\psi^{(\mathfrak{T}\mathcal{Z})}}{\delta\mathcal{Z}} - 3\frac{\delta^2\mu}{\delta\mathcal{Z}^2}\psi^{(\mathfrak{T}\mathcal{Z})} \\ +3\frac{\delta\mu}{\delta\mathcal{Z}}(\tau_g + \xi^{(\mathfrak{Q})})(\kappa_g + div\mathbf{Q}) \\ +2\mu\frac{\delta(\tau_g+\xi^{(\mathfrak{Q})})}{\delta\mathcal{Z}}(\kappa_g + div\mathbf{Q}) + \mu(\tau_g + \xi^{(\mathfrak{Q})})\frac{\delta(div\mathbf{Q}+\kappa_g)}{\delta\mathcal{Z}} \\ +\mu(\tau_g + \xi^{(\mathfrak{Q})})^2\psi^{(\mathfrak{T}\mathfrak{Q})} + \mu\psi^{(\mathfrak{T}\mathcal{Z})}(div\mathbf{Q} + \kappa_g)^2 - \mu\frac{\delta^2\psi^{(\mathfrak{T}\mathcal{Z})}}{\delta\mathcal{Z}^2} \end{pmatrix}\mathbf{T}\\ &+ \begin{pmatrix} \frac{\delta^3\mu}{\delta\mathcal{Z}^3} - 3\frac{\delta\mu}{\delta\mathcal{Z}}(\psi^{(\mathfrak{T}\mathcal{Z})})^2 - 3\frac{\delta\mu}{\delta\mathcal{Z}^*}(\tau_g + \xi^{(\mathfrak{Q})})^2 \\ -3\mu\psi^{(\mathfrak{T}\mathcal{Z})}\frac{\delta\psi^{(\mathfrak{T}\mathcal{Z})}}{\delta\mathcal{Z}} - 3\mu(\tau_g + \xi^{(\mathfrak{Q})})\frac{\delta(\tau_g+\xi^{(\mathfrak{Q})})}{\delta\mathcal{Z}} \end{pmatrix}\mathbf{Q}\\ &+ \begin{pmatrix} \begin{pmatrix} 3\frac{\delta^2\mu}{\delta\mathcal{Z}^2}(\tau_g + \xi^{(\mathfrak{Q})}) + 3\frac{\delta\mu}{\delta\mathcal{Z}}\psi^{(\mathfrak{T}\mathcal{Z})}(\kappa_g + div\mathbf{Q}) \\ +3\frac{\delta\mu}{\delta\mathcal{Z}}\frac{\delta(\tau_{\mathfrak{Q}}+\xi^{(\mathfrak{Q})})}{\delta\mathcal{Z}} \end{pmatrix} \\ +\mu\begin{pmatrix} 2(div\mathbf{Q} + \kappa_g)\frac{\delta\psi^{(\mathfrak{T}\mathcal{Z})}}{\delta\mathcal{Z}} + \psi^{(\mathfrak{T}\mathcal{Z})}\frac{\delta(div\mathbf{Q}+\kappa_g)}{\delta\mathcal{Z}} \\ -(\tau_g + \xi^{(\mathfrak{Q})})(div\mathbf{Q} + \kappa_g)^2 \end{pmatrix} \\ \begin{pmatrix} -(\tau_g + \xi^{(\mathfrak{Q})})^3 - (\psi^{(\mathfrak{T}\mathcal{Z})})^2(\tau_g + \xi^{(\mathfrak{Q})}) + \frac{\delta^2(\tau_g+\xi^{(\mathfrak{Q})})}{\delta\mathcal{Z}^2} \end{pmatrix} \end{pmatrix}\mathbf{Z}\end{aligned}$$

From (11),

$$\left.\frac{\delta\lambda}{\delta u}\right|_{u=0}\sqrt{\left\langle\left.\frac{\delta\varphi}{\delta\mathcal{Z}^*}\right|_{u=0}, \left.\frac{\delta\varphi}{\delta\mathcal{Z}^*}\right|_{u=0}\right\rangle}$$
$$+\int_0^l \left\langle\left.\frac{\delta\varphi}{\delta\mathcal{Z}^*}\right|_{u=0}, \left.\frac{\delta^2\varphi}{\delta\mathcal{Z}^*\delta u}\right|_{u=0}\right\rangle\left\langle\left.\frac{\delta\varphi}{\delta\mathcal{Z}^*}\right|_{u=0}, \left.\frac{\delta\varphi}{\delta\mathcal{Z}^*}\right|_{u=0}\right\rangle^{-1/2}\delta\mathcal{Z}^* = 0.$$

$\mathcal{I}_2(u)$ is obtained as following from (8):

$$\mathcal{I}_2(u) = \int_0^{\lambda(u)} \left\langle \frac{\delta\varphi}{\delta\mathcal{Z}^*} \wedge \frac{\delta^2\varphi}{\delta\mathcal{Z}^{*2}}, \frac{\delta^3\varphi}{\delta\mathcal{Z}^{*3}} \right\rangle \left\langle \frac{\delta\varphi}{\delta\mathcal{Z}^*}, \frac{\delta\varphi}{\delta\mathcal{Z}^*} \right\rangle^{-5/2} \delta\mathcal{Z}^*. \tag{12}$$

The arc γ is a extremal if and only if $\frac{\delta\mathcal{I}_2}{\delta u}(0) = 0$. From (12),

$$\begin{aligned}
\frac{\delta\mathcal{I}_2}{\delta u}(u) &= \frac{\delta\lambda}{\delta u}\left\{\left\langle \frac{\delta\varphi}{\delta\mathcal{Z}^*} \wedge \frac{\delta^2\varphi}{\delta\mathcal{Z}^{*2}}, \frac{\delta^3\varphi}{\delta\mathcal{Z}^{*3}} \right\rangle^2 \left\langle \frac{\delta\varphi}{\delta\mathcal{Z}^*}, \frac{\delta\varphi}{\delta\mathcal{Z}^*} \right\rangle^{-5/2}\right\}_{\mathcal{Z}^*=\lambda(u)} \\
&+ \int_0^{\lambda(u)} \left(\begin{array}{c} \left(\left\langle \frac{\delta\varphi}{\delta\mathcal{Z}^*} \wedge \frac{\delta^3\varphi}{\delta u\delta\mathcal{Z}^{*2}}, \frac{\delta^3\varphi}{\delta\mathcal{Z}^{*3}} \right\rangle + \left\langle \frac{\delta^2\varphi}{\delta u\delta\mathcal{Z}^*} \wedge \frac{\delta^2\varphi}{\delta\mathcal{Z}^{*2}}, \frac{\delta^3\varphi}{\delta\mathcal{Z}^3} \right\rangle \right. \\ \left. + \left\langle \frac{\delta\varphi}{\delta\mathcal{Z}^*} \wedge \frac{\delta^2\varphi}{\delta\mathcal{Z}^{*2}}, \frac{\delta^4\varphi}{\delta u\delta\mathcal{Z}^{*3}} \right\rangle\right) \left\langle \frac{\delta\varphi}{\delta\mathcal{Z}^*}, \frac{\delta\varphi}{\delta\mathcal{Z}^*} \right\rangle^{-5/2} \end{array} \right) \delta\mathcal{Z}^* \\
&-5\int_0^{\lambda(u)} \left\langle \frac{\delta\varphi}{\delta\mathcal{Z}^*} \wedge \frac{\delta^2\varphi}{\delta\mathcal{Z}^{*2}}, \frac{\delta^3\varphi}{\delta\mathcal{Z}^{*3}} \right\rangle \left\langle \frac{\delta\varphi}{\delta\mathcal{Z}^*}, \frac{\delta\varphi}{\delta\mathcal{Z}^*} \right\rangle^{-7/2} \left\langle \frac{\delta^2\varphi}{\delta u\delta\mathcal{Z}^*}, \frac{\delta\varphi}{\delta\mathcal{Z}^*} \right\rangle \delta\mathcal{Z}^*
\end{aligned}$$

and

$$\begin{aligned}
\frac{\delta\mathcal{I}_2(0)}{\delta u} = \int_0^l \mu \left\{ \begin{array}{c}
-2\frac{\delta^3(div\mathbf{Q}+\kappa_g)}{\delta\mathcal{Z}^3} + 6(\tau_g + \psi^{(\mathfrak{T}\mathfrak{Q})})\frac{\delta^2\psi^{(\mathfrak{T}\mathcal{Z})}}{\delta\mathcal{Z}^2} \\
+3(\kappa_g + div\mathbf{Q})^2\frac{\delta(div\mathbf{Q}+\kappa_g)}{\delta\mathcal{Z}} + 6\frac{\delta(\tau_g+\xi^{(\mathfrak{Q})})}{\delta\mathcal{Z}}\frac{\delta(\psi^{(\mathfrak{T}\mathcal{Z})})}{\delta\mathcal{Z}} \\
+3(\psi^{(\mathfrak{T}\mathcal{Z})})^2\frac{\delta(\kappa_g+div\mathbf{Q})}{\delta\mathcal{Z}} - 2(\psi^{(\mathfrak{T}\mathcal{Z})})(\tau_{\mathfrak{Q}} + \xi^{(\mathfrak{Q})})^3 \\
+2(\psi^{(\mathfrak{T}\mathcal{Z})})\frac{\delta^2(\tau_g+\xi^{(\mathfrak{Q})})}{\delta\mathcal{Z}^2} + 6(\kappa_g + div\mathbf{Q})(\tau_g + \psi^{(\mathfrak{T}\mathfrak{Q})})\frac{\delta(\tau_g+\xi^{(\mathfrak{Q})})}{\delta\mathcal{Z}} \\
+6(\tau_g + \xi^{(\mathfrak{Q})})^2\frac{\delta(div\mathbf{Q}+\kappa_g)}{\delta\mathcal{Z}} + 3(\psi^{(\mathfrak{T}\mathcal{Z})})(\tau_g + \xi^{(\mathfrak{Q})})(\kappa_g + div\mathbf{Q})^2 \\
+3(div\mathfrak{Q} + \kappa_g)^3(\tau_g + \xi^{(\mathfrak{Q})}) \\
+\psi^{(\mathfrak{T}\mathcal{Z})}\left(\begin{array}{c} (div\mathbf{Q} + \kappa_g)(l)\frac{\delta\psi^{(\mathfrak{T}\mathcal{Z})}}{\delta\mathcal{Z}}(l) + (div\mathbf{Q} + \kappa_g)^2(l)(\tau_g + \xi^{(\mathfrak{Q})})(l) \\ +(\psi^{(\mathfrak{T}\mathcal{Z})})^2(l)(\tau_g + \xi^{(\mathfrak{Q})})(l) - (\psi^{(\mathfrak{T}\mathcal{Z})})(l)\frac{\delta(\kappa_g+div\mathbf{Q})}{\delta\mathcal{Z}}(l) \end{array} \right)
\end{array} \right\} \delta\mathcal{Z}
\end{aligned}$$

$$
\begin{aligned}
&+\mu(\mathfrak{l})\left(\begin{array}{c} -(\kappa_g+div\mathbf{Q})(l)(\psi^{(\mathfrak{TZ})})^2(l)-5\frac{\delta(\kappa_g+div\mathbf{Q})}{\delta\mathcal{Z}}(l)(\tau_g+\xi^{(\mathfrak{Q})})^2(l) \\ -6(\tau_g+\xi^{(\mathfrak{Q})})(l)\frac{\delta\psi^{(\mathfrak{TZ})}}{\delta\mathcal{Z}}(l)-(\psi^{(\mathfrak{TZ})})(l)\frac{\delta(\tau_g+\xi^{(\mathfrak{Q})})}{\delta\mathcal{Z}}(l) \\ +2\frac{\delta^3(div\mathbf{Q}+\kappa_g)}{\delta\mathcal{Z}^3}(l)+(div\mathbf{Q}+\kappa_g)^3(l)\end{array}\right) \\
&+\frac{\delta\mu}{\delta\mathcal{Z}}(l)\left(\ 4(\psi^{(\mathfrak{TZ})})(l)(\tau_g+\xi^{(\mathfrak{Q})})(l)-2\frac{\delta(div\mathbf{Q}+\kappa_g)}{\delta\mathcal{Z}}(l)\ \right) \\
&+\frac{\delta^2\mu}{\delta\mathcal{Z}^2}(l)(div\mathbf{Q}+\kappa_g)(l)
\end{aligned}
$$

$\frac{\delta\mathcal{I}_2(0)}{\delta u}=0$ if and only if the arc γ must satisfies three boundary conditions and a differential equation:

$$
\begin{aligned}
&{}^{(\mathcal{Z})}(\mathfrak{BC}1)\quad (div\mathbf{Q}+\kappa_g)(l)=0 \\
&{}^{(\mathcal{Z})}(\mathfrak{BC}2)\quad 2(\psi^{(\mathfrak{TZ})})(l)(\tau_g+\xi^{(\mathfrak{Q})})(l)=\tfrac{\delta(div\mathbf{Q}+\kappa_g)}{\delta\mathcal{Z}}(l) \\
&{}^{(\mathcal{Z})}(\mathfrak{BC}3)\quad \left\{\begin{array}{c} 2\frac{\delta^2(div\mathbf{Q}+\kappa_g)}{\delta\mathcal{Z}^2}(l)=(div\mathbf{Q}+\kappa_g)(l)(\psi^{(\mathfrak{TZ})})^2(l) \\ +5\frac{\delta(div\mathbf{Q}+\kappa_g)}{\delta\mathcal{Z}}(l)(\tau_g+\xi^{(\mathfrak{Q})})^2(l) \\ +6(\tau_g+\xi^{(\mathfrak{Q})})(l)\frac{\delta\psi^{(\mathfrak{TZ})}}{\delta\mathcal{Z}}(l) \\ +(\psi^{(\mathfrak{TZ})})(l)\frac{\delta(\tau_g+\xi^{(\mathfrak{Q})})}{\delta\mathcal{Z}}(l)-(div\mathbf{Q}+\kappa_g)^3(l)\end{array}\right. \\
&{}^{(\mathcal{Z})}(\mathfrak{DE})\left\{\begin{array}{c} -2\frac{\delta^3(div\mathbf{Q}+\kappa_g)}{\delta\mathcal{Z}^3}+6(\tau_g+\xi^{(\mathfrak{Q})})\frac{\delta^2\psi^{(\mathfrak{TZ})}}{\delta\mathcal{Z}^2}+3(div\mathbf{Q}+\kappa_g)^2\frac{\delta(div\mathbf{Q}+\kappa_g)}{\delta\mathcal{Z}} \\ +3(\psi^{(\mathfrak{TZ})})^2\frac{\delta(div\mathbf{Q}+\kappa_g)}{\delta\mathcal{Z}}-2(\psi^{(\mathfrak{TZ})})(\tau_g+\xi^{(\mathfrak{Q})})^3+2(\psi^{(\mathfrak{TZ})})\frac{\delta^2(\tau_g+\xi^{(\mathfrak{Q})})}{\delta\mathcal{Z}^2} \\ +6(div\mathbf{Q}+\kappa_g)(\tau_g+\xi^{(\mathfrak{Q})})\frac{\delta(\tau_g+\xi^{(\mathfrak{Q})})}{\delta\mathcal{Z}}+6(\tau_g+\xi^{(\mathfrak{Q})})^2\frac{\delta(div\mathbf{Q}+\kappa_g)}{\delta\mathcal{Z}} \\ +3(\psi^{(\mathfrak{TZ})})(\tau_g+\xi^{(\mathfrak{Q})})(div\mathbf{Q}+\kappa_g)^2 \\ +3(div\mathbf{Q}+\kappa_g)^3(\tau_g+\xi^{(\mathfrak{Q})})+6\frac{\delta(\tau_g+\xi^{(\mathfrak{Q})})}{\delta\mathcal{Z}}\frac{\delta(\psi^{(\mathfrak{TZ})})}{\delta\mathcal{Z}} \\ +\psi^{(\mathfrak{TZ})}\left(\begin{array}{c}(div\mathbf{Q}+\kappa_g)(l)\frac{\delta\psi^{(\mathfrak{TZ})}}{\delta\mathcal{Z}}(l)+(div\mathbf{Q}+\kappa_g)^2(l)(\tau_g+\xi^{(\mathfrak{Q})})(l) \\ +(\psi^{(\mathfrak{TZ})})^2(l)(\tau_g+\xi^{(\mathfrak{Q})})(l)-(\psi^{(\mathfrak{TZ})})(l)\frac{\delta(div\mathbf{Q}+\kappa_g)}{\delta\mathcal{Z}}(l)\end{array}\right)=0\end{array}\right.
\end{aligned}
$$

3 Conformal Magnetic Curves for the Darboux Frame in $\mathbb{R}^3$

The conformable derivative equations of the Darboux frame in $\mathbb{R}^3$ are introduced:

$$\begin{pmatrix} \mathcal{D}_{\alpha;\mathcal{T}}\mathbf{T}(\mathcal{T}) \\ \mathcal{D}_{\alpha;\mathcal{T}}\mathbf{Q}(\mathcal{T}) \\ \mathcal{D}_{\alpha;\mathcal{T}}\mathbf{Z}(\mathcal{T}) \end{pmatrix} = \begin{pmatrix} 0 & \mathfrak{T}^{1-\alpha}\kappa_g & \mathfrak{T}^{1-\alpha}\kappa_n \\ -\mathfrak{T}^{1-\alpha}\kappa_g & 0 & \mathfrak{T}^{1-\alpha}\tau_g \\ -\mathfrak{T}^{1-\alpha}\kappa_n & -\mathfrak{T}^{1-\alpha}\tau_g & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{Q} \\ \mathbf{Z} \end{pmatrix} \tag{13}$$

Here, the conformal geodesic curvature is $\mathfrak{T}^{1-\alpha}\kappa_g$, the conformable normal curvature is $\mathfrak{T}^{1-\alpha}\kappa_n$ and the conformal geodesic torsion is $\mathfrak{T}^{1-\alpha}\tau_g$. The conformal tangent vector is $\mathbf{T}(\mathcal{T}) = \mathcal{D}_{\alpha;\mathcal{T}}\gamma(\mathcal{T})\mathcal{T}^{\alpha-1}$, the conformal vector field $\mathbf{Z}(\mathcal{T})$ and $\mathbf{Q}(\mathcal{T}) = \mathbf{Z}(\mathcal{T}) \times \mathbf{T}(\mathcal{T})$.

The 1st conformal new frame is introduced:

$$U_1^{\alpha} = \mathbf{T} \tag{14}$$

$$U_2^{\alpha} = \frac{1}{\sqrt{2}}(\mathbf{Q} + i\mathbf{Z})e^{i\mathbb{I}_{\alpha}^{x}(\mathfrak{T}^{1-\alpha}\tau_g)} \tag{15}$$

$$U_2^{*\alpha} = \frac{1}{\sqrt{2}}(\mathbf{Q} - i\mathbf{Z})e^{-i\mathbb{I}_{\alpha}^{x}(\mathfrak{T}^{1-\alpha}\tau_g)} \tag{16}$$

The 1st conformal map connected with the conformal nonlinear Schrödinger equation (CNLSE) is introduced:

$$\wp_1^{\alpha} = \frac{1}{\sqrt{2}}(\mathfrak{T}^{1-\alpha}\kappa_g + i\mathfrak{T}^{1-\alpha}\kappa_n)e^{i\mathbb{I}_{\alpha}^{x}(\mathfrak{T}^{1-\alpha}\tau_g)}$$

The 2nd conformal new frame is given by

$$V_1^{\alpha} = \mathbf{Q} \tag{17}$$

$$V_2^{\alpha} = \frac{1}{\sqrt{2}}(\mathbf{T} + i\mathbf{Z})e^{i\mathbb{I}_{\alpha}^{x}(\mathfrak{T}^{1-\alpha}\kappa_n)} \tag{18}$$

$$V_2^{*\alpha} = \frac{1}{\sqrt{2}}(\mathbf{T} - i\mathbf{Z})e^{-i\mathbb{I}_{\alpha}^{x}(\mathfrak{T}^{1-\alpha}\kappa_n)} \tag{19}$$

The 2nd conformal map connected with CLNSE is introduced

$$\wp_2^{\alpha} = \frac{1}{\sqrt{2}}(i\mathfrak{T}^{1-\alpha}\tau_g - \mathfrak{T}^{1-\alpha}\kappa_g)e^{i\mathbb{I}_{\alpha}^{x}(\mathfrak{T}^{1-\alpha}\kappa_n)}$$

The 3rd conformal new frame is given by

$$W_1^{\alpha} = \mathbf{Z} \tag{20}$$

$$W_2^{\alpha} = \frac{1}{\sqrt{2}}(\mathbf{T} + i\mathbf{Q})e^{\,i\mathbb{I}_{\alpha}^{x}(\mathfrak{T}^{1-\alpha}\kappa_g)} \tag{21}$$

$$W_2^{*\alpha} = \frac{1}{\sqrt{2}}(\mathbf{T} - i\mathbf{Q})e^{\,-i\mathbb{I}_{\alpha}^{x}(\mathfrak{T}^{1-\alpha}\kappa_g)} \tag{22}$$

The 3rd conformal map connected with CLNSE is introduced

$$\wp_3^{\alpha} = \frac{1}{\sqrt{2}}(\mathfrak{T}^{1-\alpha}\kappa_n + i\mathfrak{T}^{1-\alpha}\tau_g)e^{\,i\mathbb{I}_{\alpha}^{x}(\mathfrak{T}^{1-\alpha}\kappa_g)}$$

3.1 T-conformal Magnetic Curves for Darboux Frame in $\mathbb{R}^3$

If the conformal Lorentz force equation is satisfied

$$\mathcal{D}_{\alpha;\mathcal{T}}\mathbf{T} = {}^{c}\Phi(\mathbf{T}) = \mathfrak{T}^{1-\alpha}\kappa_g\mathbf{Q} + \mathfrak{T}^{1-\alpha}\kappa_n\mathbf{Z} \tag{23}$$

then, the curve is called **T**-conformal magnetic curve for Darboux frame in $\mathbb{R}^3$.

Theorem 1. *Lorentz conformal force equations of the Darboux frame for* **T***-conformal magnetic curve in* $\mathbb{R}^3$ *are given by*

$$\begin{bmatrix} {}^{c}\Phi(\mathbf{T}) \\ {}^{c}\Phi(\mathbf{Q}) \\ {}^{c}\Phi(\mathbf{Z}) \end{bmatrix} = \begin{bmatrix} 0 & \mathfrak{T}^{1-\alpha}\kappa_g & \mathfrak{T}^{1-\alpha}\kappa_n \\ -\mathfrak{T}^{1-\alpha}\kappa_g & 0 & \omega_1 \\ -\mathfrak{T}^{1-\alpha}\kappa_n & 0 & -\omega_1 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{Q} \\ \mathbf{Z} \end{bmatrix} \tag{24}$$

Proof. Consider **T**-conformable magnetic curve for Darboux frame in $\mathbb{R}^3$. Hence,

$${}^{c}\Phi(\mathbf{T}) = \mathfrak{T}^{1-\alpha}\kappa_g\mathbf{Q} + \mathfrak{T}^{1-\alpha}\kappa_n\mathbf{Z} \tag{25}$$

Consider

$${}^{c}\Phi(\mathbf{Q}) = \mathfrak{x}_1\mathbf{T} + \mathfrak{y}_1\mathbf{Q} + \mathfrak{z}_1\mathbf{Z} \tag{26}$$

From Eqs.(25) and (26), the followings are obtained:

$$\langle {}^{c}\Phi(\mathbf{Q}), \mathbf{T}\rangle = \mathfrak{x}_1 = -\langle {}^{c}\Phi(\mathbf{T}), \mathbf{Q}\rangle = -\mathfrak{T}^{1-\alpha}\kappa_g \tag{27}$$

$$\langle {}^{c}\Phi(\mathbf{Q}), \mathbf{Q}\rangle = \mathfrak{y}_1 = 0 \tag{28}$$

$$\langle {}^{c}\Phi(\mathbf{Q}), \mathbf{Z}\rangle = \mathfrak{z}_1 = \omega_1 \tag{29}$$

With the aid of Eqs.(26), (27), (28) and (29), it can be obtained

$${}^{c}\Phi(\mathbf{Q}) = -\mathcal{T}^{\mathbf{1}-\alpha}\kappa_{\mathbf{g}}\mathbf{T} + \omega_1\mathbf{Z}$$

Additionally,

$${}^{c}\Phi(\mathbf{Z}) = \mathfrak{x}_2\mathbf{T} + \mathfrak{y}_2\mathbf{Q} + \mathfrak{z}_2\mathbf{Z} \tag{30}$$

From Eq.(30)

$$\langle {}^{c}\Phi(\mathbf{Z}), \mathbf{T}\rangle = \mathfrak{x}_2 = -\langle {}^{c}\Phi(\mathbf{T}), \mathbf{Z}\rangle = -\mathfrak{T}^{1-\alpha}\kappa_n \tag{31}$$

$$\langle {}^{c}\Phi(\mathbf{Z}), \mathbf{Q}\rangle = \mathfrak{y}_2 = -\omega_1 \tag{32}$$

$$\langle {}^{c}\Phi(\mathbf{Z}), \mathbf{Z}\rangle = \mathfrak{z}_2 = 0 \tag{33}$$

From Eqs.(30), (31), (32) and (33),

$${}^{c}\Phi(\mathbf{Z}) = -\mathcal{T}^{1-\alpha}\kappa_n\mathbf{T} - \omega_1\mathbf{Q}$$

□

3.2 Q-conformal Magnetic Curves for the Darboux Frame in $\mathbb{R}_1^3$

If Lorentz conformal force equation is satisfied

$$\mathcal{D}_{\alpha;\mathcal{T}}\mathbf{Q} = {}^{c}\Phi(\mathbf{Q}) = -\mathfrak{T}^{1-\alpha}\kappa_g\mathbf{T} + \mathfrak{T}^{1-\alpha}\tau_g\mathbf{Z} \tag{34}$$

then, the curve is called **Q**-conformable magnetic curve for Darboux frame in $\mathbb{R}^3$.

Theorem 2. *Lorentz conformal force equations of the Darboux frame for* **Q***-conformal magnetic curve in* $\mathbb{R}^3$ *are given by*

$$\begin{bmatrix} {}^{c}\Phi(\mathbf{T}) \\ {}^{c}\Phi(\mathbf{Q}) \\ {}^{c}\Phi(\mathbf{Z}) \end{bmatrix} = \begin{bmatrix} 0 & \mathfrak{T}^{1-\alpha}\kappa_g & \omega_2 \\ -\mathfrak{T}^{1-\alpha}\kappa_g & 0 & \mathfrak{T}^{1-\alpha}\tau_g \\ -\omega_2 & -\mathfrak{T}^{1-\alpha}\tau_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{Q} \\ \mathbf{Z} \end{bmatrix} \tag{35}$$

Proof. Consider **Q**-conformable magnetic curve for Darboux frame in $\mathbb{R}^3$. Then,

$${}^{c}\Phi(\mathbf{Q}) = -\mathfrak{T}^{1-\alpha}\kappa_g\mathbf{T} + \mathfrak{T}^{1-\alpha}\tau_g\mathbf{Z}$$

Moreover,

$${}^{c}\Phi(\mathbf{T}) = \mathfrak{x}_3\mathbf{T} + \mathfrak{y}_3\mathbf{Q} + \mathfrak{z}_3\mathbf{Z} \tag{36}$$

From Eqs.(34) and (36), the followings are obtained:

$$\langle {}^{c}\Phi(\mathbf{T}), \mathbf{T}\rangle = \mathfrak{x}_3 = 0 \tag{37}$$
$$\langle {}^{c}\Phi(\mathbf{T}), \mathbf{Q}\rangle = \mathfrak{y}_3 = \mathfrak{T}^{1-\alpha}\kappa_g \tag{38}$$
$$\langle {}^{c}\Phi(\mathbf{T}), \mathbf{Z}\rangle = \mathfrak{z}_3 = \omega_2 \tag{39}$$

From Eqs.(36), (37), (38) and (39),

$${}^{c}\Phi(\mathbf{T}) = -\mathcal{T}^{\mathbf{1}-\alpha}\kappa_{\mathbf{g}}\mathbf{Q} + \omega_2\mathbf{Z} \tag{40}$$

And

$${}^{c}\Phi(\mathbf{Z}) = \mathfrak{x}_4\mathbf{T} + \mathfrak{y}_4\mathbf{Q} + \mathfrak{z}_4\mathbf{Z} \tag{41}$$

From Eqs.(34) and (41),

$$\langle {}^{c}\Phi(\mathbf{Z}), \mathbf{T}\rangle = \mathfrak{x}_4 = -\langle {}^{c}\Phi(\mathbf{T}), \mathbf{Z}\rangle = -\omega_2 \tag{42}$$
$$\langle {}^{c}\Phi(\mathbf{Z}), \mathbf{Q}\rangle = \mathfrak{y}_4 = -\mathcal{T}^{1-\alpha}\tau_g \tag{43}$$
$$\langle {}^{c}\Phi(\mathbf{Z}), \mathbf{Z}\rangle = \mathfrak{z}_4 = 0 \tag{44}$$

From Eqs.(42), (43) and (44),

$${}^{c}\Phi(\mathbf{Z}) = -\omega_2\mathbf{T} - \mathcal{T}^{1-\alpha}\tau_g\mathbf{Q}$$

□

3.3 Z-conformal Magnetic Curves for Darboux Frame in $\mathbb{R}^3$

If Lorentz conformal force equation for the Darboux frame is satisfied

$$\mathcal{D}_{\alpha;\mathcal{T}}\mathbf{Z} = {}^{c}\Phi(\mathbf{Z}) = -\mathfrak{T}^{1-\alpha}\kappa_n\mathbf{T} + \mathfrak{T}^{1-\alpha}\tau_g\mathbf{Q} \tag{45}$$

then, the curve is called $\mathbf{Z}$-conformal magnetic curve for the Darboux frame in $\mathbb{R}^3$.

Theorem 3. *Lorentz conformal force equations of the Darboux frame for* $\mathbf{Z}$*-conformal magnetic curve in* $\mathbb{R}^3$ *are given by*

$$\begin{bmatrix} {}^{c}\Phi(\mathbf{T}) \\ {}^{c}\Phi(\mathbf{Q}) \\ {}^{c}\Phi(\mathbf{Z}) \end{bmatrix} = \begin{bmatrix} 0 & \omega_4 & \mathfrak{T}^{1-\alpha}\kappa_n \\ -\omega_4 & 0 & -\mathfrak{T}^{1-\alpha}\tau_g \\ -\mathfrak{T}^{1-\alpha}\kappa_n & \mathfrak{T}^{1-\alpha}\tau_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{Q} \\ \mathbf{Z} \end{bmatrix}$$

Proof. Consider $\mathbf{Z}$-conformable magnetic curve for the Darboux frame in $\mathbb{R}^3$. Then,

$${}^{c}\Phi(\mathbf{Q}) = -\mathfrak{T}^{1-\alpha}\kappa_n\mathbf{T} + \mathfrak{T}^{1-\alpha}\tau_g\mathbf{Q}$$

Moreover,

$${}^{c}\Phi(\mathbf{T}) = \mathfrak{x}_5\mathbf{T} + \mathfrak{y}_5\mathbf{Q} + \mathfrak{z}_5\mathbf{Z} \tag{46}$$

From Eqs.(45) and (46), the followings are obtained:

$$\langle {}^{c}\Phi(\mathbf{T}), \mathbf{T}\rangle = \mathfrak{x}_5 = 0 \tag{47}$$
$$\langle {}^{c}\Phi(\mathbf{T}), \mathbf{Q}\rangle = \mathfrak{y}_5 = \omega_3 \tag{48}$$
$$\langle {}^{c}\Phi(\mathbf{T}), \mathbf{Z}\rangle = \mathfrak{z}_5 = \mathfrak{T}^{1-\alpha}\kappa_n \tag{49}$$

From Eqs.(46), (47), (48) and (49),

$${}^{c}\Phi(\mathbf{T}) = \omega_3\mathbf{Q} + \mathfrak{T}^{1-\alpha}\kappa_n\mathbf{Z} \tag{50}$$

And

$${}^{c}\Phi(\mathbf{Q}) = \mathfrak{x}_6\mathbf{T} + \mathfrak{y}_6\mathbf{Q} + \mathfrak{z}_6\mathbf{Z} \tag{51}$$

From Eqs.(45) and (51),

$$\langle {}^{c}\Phi(\mathbf{Q}), \mathbf{T}\rangle = \mathfrak{x}_6 = -\omega_3 \tag{52}$$
$$\langle {}^{c}\Phi(\mathbf{Q}), \mathbf{Q}\rangle = \mathfrak{y}_6 = 0 \tag{53}$$
$$\langle {}^{c}\Phi(\mathbf{Q}), \mathbf{Z}\rangle = \mathfrak{z}_6 = -\mathfrak{T}^{1-\alpha}\tau_g \tag{54}$$

From Eqs.(51), (52), (53) and (54),

$${}^{c}\Phi(\mathbf{Q}) = -\omega_3\mathbf{T} - \mathcal{T}^{1-\alpha}\tau_g\mathbf{Q}$$

□

3.4 The Conformal Differential Formula of the Conformal Map $\wp_1^\alpha$ in $\mathbb{R}^3$

The vector field $\mathbf{X}_1 = f_1\mathbf{T} + g_1\mathbf{Q} + h_1\mathbf{Z}$ satisfies the conformal arclength preserving condition:

$$\left\langle \mathcal{D}_{\alpha;\mathcal{T}}\mathbf{X}_1, \mathbf{T}\right\rangle = 0 \Rightarrow \mathcal{D}_{\alpha;\mathcal{T}} f_1 = g_1\mathfrak{T}^{1-\alpha}\kappa_g + h_1\mathfrak{T}^{1-\alpha}\kappa_n$$

The conformal normalization operator associated with the CNLSE for the 1st class is given by

$$C\mathcal{N}_1\mathbf{X}_1 = \left(\mathbb{I}_\alpha^x(\mathfrak{T}^{1-\alpha}g_1\kappa_g + \mathfrak{T}^{1-\alpha}h_1\kappa_n\right)\mathbf{T} + g_1\mathbf{Q}+h_1\mathbf{Z} \tag{55}$$

The conformal linear recursion operator connected with the CNLSE for the 1st class is

$$C\mathcal{R}_1^1\mathbf{X}_1 = -C\mathcal{N}_1(\mathcal{D}_{\alpha;\mathcal{T}}^2\mathbf{T}\times D_{\alpha;\mathcal{T}}\mathbf{X}_1) \tag{56}$$

The conformal differential formula of $\wp_1^\alpha$ assocated with the CNLSE for 1st class in $\mathbb{R}^3$ is

$$\mathcal{D}\wp_1^\alpha(\mathbf{X}_1) = \left\langle \mathbf{Q}+i\mathbf{Z}, \mathbf{T}\times\mathcal{D}_{\alpha;\mathcal{T}}(C\mathcal{R}_1^1\mathbf{X}_1)\right\rangle \varsigma_1^\alpha, \quad \varsigma_1^\alpha = \frac{1}{\sqrt{2}}e^{i\mathbb{I}_\alpha^x(\mathfrak{T}^{1-\alpha}\tau_g)} \tag{57}$$

Consider the conformal flow equation connected with the CNLSE for the curve β_1

$$\mathbf{X}_1 = \mathcal{D}_{\alpha;t}\beta_1 = \mathcal{D}_{\alpha;\mathcal{T}}\beta_1\times\mathcal{D}_{\alpha;\mathcal{T}}^2\beta_1 = -\mathfrak{T}^{1-\alpha}\kappa_n\mathbf{Q} + \mathfrak{T}^{1-\alpha}\kappa_g\mathbf{Z}$$

$$\begin{aligned}\mathbf{T}\times C_{\alpha;\mathcal{T}}\mathbf{X}_1 &= -(\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_g) - \mathfrak{T}^{2(1-\alpha)}\kappa_n\tau_g)\mathbf{Q}\\ &\quad -(\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_n) + \mathfrak{T}^{2(1-\alpha)}\kappa_g\tau_g)\mathbf{Z}\end{aligned}$$

The conformal normalization for the 1st class is given by

$$\begin{aligned}C\mathcal{N}_1\mathbf{X}_1 &= \left(\mathbb{I}_\alpha^x(\mathfrak{T}^{1-\alpha}\kappa_g\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_g) + \mathfrak{T}^{1-\alpha}\kappa_n\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_n)\right)\mathbf{T}\\ &\quad -(\mathfrak{T}^{2(1-\alpha)}\kappa_n\tau_g - \mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_g))\mathbf{Q}\\ &\quad +(\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_n) + \mathfrak{T}^{2(1-\alpha)}\kappa_g\tau_g)\mathbf{Z}\end{aligned}$$

$$\begin{aligned}\mathcal{D}\wp_1^\alpha(\mathbf{X}_1) &= -(\mathcal{D}_{\alpha;\mathcal{T}}^2(\mathfrak{T}^{1-\alpha}\kappa_n) + \mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{2(1-\alpha)}\kappa_g\tau_g) + \mathfrak{T}^{1-\alpha}\tau_g\mathcal{D}_{\alpha;T}(\mathfrak{T}^{1-\alpha}\kappa_g)\\ &\quad -\mathfrak{T}^{3(1-\alpha)}\kappa_n{\tau_g}^2 + \frac{1}{2}\mathfrak{T}^{3(1-\alpha)}\kappa_n({\kappa_g}^2 + {\kappa_n}^2))\\ &\quad +i(\mathcal{D}_{\alpha;\mathcal{T}}^2(\mathfrak{T}^{1-\alpha}\kappa_g) - \mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{2(1-\alpha)}\kappa_n\tau_g) - \mathfrak{T}^{1-\alpha}\tau_g\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_n)\\ &\quad -\mathfrak{T}^{3(1-\alpha)}\kappa_g{\tau_g}^2 + \frac{1}{2}\mathfrak{T}^{3(1-\alpha)}\kappa_g({\kappa_g}^2 + {\kappa_n}^2))\varsigma_1^\alpha\end{aligned} \tag{58}$$

From Eqs.(57) and (58), the CNLSE is obtained:

$$\mathcal{D}_{\alpha;t}\wp_1^\alpha = i\mathcal{D}_{\alpha;\mathcal{T}}^2\wp_1^\alpha + i(\wp_1^{\alpha*}\wp_1^\alpha)\wp_1^\alpha$$

3.5 The Conformal Differential Formula of the Map $\wp_2^\alpha$ for $\{\mathbf{T},\mathbf{Q},\mathbf{Z}\}$ in $\mathbb{R}^3$

$\mathbf{X}_2 = f_2\mathbf{T} + g_2\mathbf{Q} + h_2\mathbf{Z}$ satisfies the conformal arclength preserving condition:

$$\left\langle\mathcal{D}_{\alpha;\mathcal{T}}\mathbf{X}_2, \mathbf{Q}\right\rangle = 0 \Rightarrow \mathcal{D}_{\alpha;\mathcal{T}}g_2 = \mathfrak{T}^{1-\alpha}\tau_g h_2 - \mathfrak{T}^{1-\alpha}f_2\kappa_g$$

The conformal normalization with CNLSE for the 2nd class is given by

$$C\mathcal{N}_2\mathbf{X}_2 = \left(\mathbb{I}_\alpha^x(\mathfrak{T}^{1-\alpha}\tau_g h_2 - \mathfrak{T}^{1-\alpha} f_2\kappa_g\right)\mathbf{Q} + f_2\mathbf{T}+h_2\mathbf{Z}$$

The conformal linear recursion operator connected with the CNLSE for the 2nd class is given by

$$C\mathcal{R}_2^1\mathbf{X}_2 = C\mathcal{N}_2(\mathbf{Q}\times\mathcal{D}_{\alpha;T}\mathbf{X}_2)$$

The conformal differential formula of the conformal map $\wp_2^\alpha$ with the CNLSE is given by

$$\mathcal{D}\wp_2^\alpha(\mathbf{X}_2) = \left\langle \mathbf{T}+i\mathbf{Z}, \mathbf{Q}\times\mathcal{D}_{\alpha;\mathcal{T}}({}^c\mathcal{R}_2^{(1)}\mathbf{X}_2)\right\rangle\ \varsigma_2^\alpha, \quad \varsigma_2^\alpha = \frac{1}{\sqrt{2}}e^{i\mathbb{I}_\alpha^x(\mathfrak{T}^{1-\alpha}\kappa_n)} \tag{59}$$

If the conformal differential formula (59) is applied to $\mathbf{X}_2$ for the 2nd class connected with the CNLSE with the curve β_2, then

$$\mathbf{X}_2 = \mathcal{D}_{\alpha;u}\beta_2 = \mathcal{D}_{\alpha;\mathcal{T}}\beta_2 \times \mathcal{D}^2_{\alpha;\mathcal{T}}\beta_2 = \mathfrak{T}^{1-\alpha}\tau_g\mathbf{T} + \mathfrak{T}^{1-\alpha}\kappa_g\mathbf{Z}$$

$$\mathbf{Q}\times\mathcal{D}_{\alpha;\mathcal{T}}\mathbf{X}_2 = (\mathfrak{T}^{2(1-\alpha)}\kappa_n\tau_g + \mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_g)\mathbf{T} + (\mathfrak{T}^{2(1-\alpha)}\kappa_g\kappa_n - \mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\tau_g))\mathbf{Z}$$

The conformal linear normalization operator for $\{\mathbf{T}, \mathbf{Q}, \mathbf{Z}\}$ connected with the CNLSE is given by

$$\begin{aligned}
C\mathcal{N}_2(\mathbf{Q}\times\mathcal{D}_{\alpha;\mathcal{T}}\mathbf{X}_2) &= -\left(\mathbb{I}_\alpha^x(\mathfrak{T}^{1-\alpha}\kappa_g\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_g) + \mathfrak{T}^{1-\alpha}\tau_g\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha)}\tau_g)\right)\mathbf{Q} \\
&\quad +(\mathfrak{T}^{2(1-\alpha)}\kappa_n\tau_g + \mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_g))\mathbf{T} + (\mathfrak{T}^{2(1-\alpha)}\kappa_g\kappa_n - \mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\tau_g))\mathbf{Z} \\
C\mathcal{R}_2^{(1)}\mathbf{X}_2 &= -\frac{1}{2}\mathfrak{T}^{2(1-\alpha)}({\kappa_g}^2 + {\tau_g}^2)\mathbf{Q} + (\mathfrak{T}^{2(1-\alpha)}\kappa_n\tau_g + \mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_g))\mathbf{T} \\
&\quad +(\mathfrak{T}^{2(1-\alpha)}\kappa_g\kappa_n - \mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\tau_g))\mathbf{Z}
\end{aligned}$$

$$\begin{aligned}
\mathbf{Q}\times\mathcal{D}_{\alpha;\mathcal{T}}(C\mathcal{R}_2^{(1)}\mathbf{X}_2) &= -(\mathcal{D}^2_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_g) + \mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{2(1-\alpha)}\kappa_n\tau_g) \\
&\quad +\mathfrak{T}^{1-\alpha}\kappa_n\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\tau_g) - (\mathfrak{T}^{3(1-\alpha)}{\kappa_n}^2\kappa_g) \\
&\quad +\frac{1}{2}\mathfrak{T}^{3(1-\alpha)}\kappa_g({\kappa_g}^2 + {\tau_g}^2))\mathbf{Z} \\
&\quad +(\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{2(1-\alpha)}\kappa_g\kappa_n) - \mathcal{D}^2_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\tau_g) \\
&\quad +\mathfrak{T}^{1-\alpha}\kappa_n\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_g) + \mathfrak{T}^{3(1-\alpha)}\kappa_n^2\tau_g \\
&\quad -\frac{1}{2}\mathfrak{T}^{3(1-\alpha)}\tau_g({\kappa_g}^2 + {\tau_g}^2))\mathbf{T}
\end{aligned}$$

The conformal differential formula for $\mathbf{X}_2$ connected with the CNLSE for $\{\mathbf{T}, \mathbf{Q}, \mathbf{Z}\}$ for 2nd class in $\mathbb{R}^3$ is derived:

$$\begin{aligned}
\mathcal{D}\wp_2^\alpha(\mathbf{X}_2) &= (-\mathcal{D}^2_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\tau_g) + \mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{2(1-\alpha)}\kappa_g\kappa_n) \\
&\quad +\mathfrak{T}^{1-\alpha}\kappa_n\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_g) + \mathfrak{T}^{3(1-\alpha)}\kappa_n^2\tau_g - \frac{1}{2}\mathfrak{T}^{3(1-\alpha)}\tau_g({\kappa_g}^2 + {\tau_g}^2)) \\
&\quad -i(\mathcal{D}^2_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_g) + \mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{2(1-\alpha)}\kappa_n\tau_g)
\end{aligned} \tag{60}$$

$$+\mathfrak{T}^{1-\alpha}\kappa_n\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\tau_g)-\mathfrak{T}^{3(1-\alpha)}\kappa_n^2\kappa_g+\frac{1}{2}\mathfrak{T}^{3(1-\alpha)}\kappa_g({\kappa_g}^2+{\tau_g}^2))\varsigma_2^\alpha$$

From Eqs.(59), (60), the CNLSE is obtained:

$$\mathcal{D}_{\alpha;t}\wp_2^\alpha=i\mathcal{D}_{\alpha;\mathcal{T}}^2\wp_2^\alpha+\frac{i}{2}(\wp_2^{\alpha*}\wp_2^\alpha)\wp_2^\alpha$$

3.6 The Conformal Differential Formula of the Map $\wp_3^\alpha$

$\mathbf{X}_3=f_3\mathbf{T}+g_3\mathbf{Q}+h_3\mathbf{Z}$ satisfies the conformable arclength preserving condition

$$\left\langle\mathcal{D}_{\alpha;\mathcal{T}}\mathbf{X}_3,\mathbf{Z}\right\rangle=0\Rightarrow\mathcal{D}_{\alpha;\mathcal{T}}h_3=\mathfrak{T}^{1-\alpha}\kappa_nf_3-\mathfrak{T}^{1-\alpha}g_3\tau_g$$

The conformal normalization with the CNLSE for the 3nd class is

$$C\mathcal{N}_3\mathbf{X}_3=\mathbb{I}_\alpha^x(\mathfrak{T}^{1-\alpha}\kappa_nf_3-\mathfrak{T}^{1-\alpha}g_3\tau_g)\mathbf{Z}+f_3\mathbf{T}+g_3\mathbf{Q}$$

The conformal linear recursion operator connected with the CNLSE for the 3rd class is

$$C\mathcal{R}_3^1\mathbf{X}_3=C\mathcal{N}_3(\mathbf{Z}\times\mathcal{D}_{\alpha;T}\mathbf{X}_3)$$

The conformal differential formula of the conformable map $\wp_3^\alpha$ with the CNLSE is given by

$$\mathcal{D}\wp_3^\alpha(\mathbf{X}_3)=\left\langle\mathbf{T}+i\mathbf{Q},\mathbf{Z}\times\mathcal{D}_{\alpha;\mathcal{T}}(C\mathcal{R}_3^{(1)}\mathbf{X}_3)\right\rangle\varsigma_3^\alpha,\quad\varsigma_3^\alpha=\frac{1}{\sqrt{2}}e^{i\mathbb{I}_\alpha^x(\mathfrak{T}^{1-\alpha}\kappa_g\tau_g)}\tag{61}$$

If the conformal differential formula (61) is applied to $\mathbf{X}_3$ for the 3rd class connected with CNLSE with the curve β_3, then

$$\mathbf{X}_3=\mathcal{D}_{\alpha;u}\beta_3=\mathcal{D}_{\alpha;\mathcal{T}}\beta_3\times\mathcal{D}_{\alpha;\mathcal{T}}^2\beta_3=\mathfrak{T}^{1-\alpha}\tau_g\mathbf{T}-\mathfrak{T}^{1-\alpha}\kappa_n\mathbf{Q}$$

$$\mathbf{Z}\times\mathcal{D}_{\alpha;\mathcal{T}}\mathbf{X}_3=(\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_n)-\mathfrak{T}^{2(1-\alpha)}\kappa_g\tau_g)\mathbf{T}+(\mathfrak{T}^{2(1-\alpha)}\kappa_g\kappa_n+\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\tau_g))\mathbf{Q}$$

The conformal linear normalization connected with CNLSE is given by

$$\begin{aligned}C\mathcal{R}_3^{(1)}\mathbf{X}_3 &= -\frac{1}{2}\mathfrak{T}^{2(1-\alpha)}({\kappa_n}^2+{\tau_g}^2)\mathbf{Z}-(\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_n)-\mathfrak{T}^{2(1-\alpha)}\kappa_g\tau_g)\mathbf{T}\\ &\quad-(\mathfrak{T}^{2(1-\alpha)}\kappa_g\kappa_n+\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\tau_g))\mathbf{Q}\end{aligned}$$

$$\begin{aligned}\mathbf{Z}\times\mathcal{D}_{\alpha;\mathcal{T}}(C\mathcal{R}_3^{(1)}\mathbf{X}_3) &= (\mathcal{D}_{\alpha;\mathcal{T}}^2(\mathfrak{T}^{1-\alpha}\tau_g)+\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{2(1-\alpha)}\kappa_g\kappa_n)\\ &\quad+\mathfrak{T}^{1-\alpha}\kappa_g\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_n)-\mathfrak{T}^{3(1-\alpha)}\tau_g\kappa_g^2\\ &\quad+\frac{1}{2}\mathfrak{T}^{3(1-\alpha)}\tau_g(\kappa_n^2+\tau_g^2))\mathbf{T}\\ &\quad+(\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{2(1-\alpha)}\kappa_g\tau_g)-\mathcal{D}_{\alpha;\mathcal{T}}^2(\mathfrak{T}^{1-\alpha}\kappa_n)\\ &\quad+\mathfrak{T}^{1-\alpha}\kappa_g\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\tau_g)+\mathfrak{T}^{3(1-\alpha)}\kappa_g^2\kappa_n\\ &\quad-\frac{1}{2}\mathfrak{T}^{3(1-\alpha)}\kappa_n({\kappa_n^2}+{\tau_g}^2))\mathbf{Q}\end{aligned}$$

The conformal differential formula connected with the CNLSE for the 3rd class is given by

$$\begin{aligned} {}^{c}\mathcal{D}\wp_3^{\alpha}(\mathbf{X}_3) &= -(\mathcal{D}^2_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\tau_g) + \mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{2(1-\alpha)}\kappa_g\kappa_n) \\ &+\mathfrak{T}^{1-\alpha}\kappa_g\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_n) - \mathfrak{T}^{3(1-\alpha)}\tau_g\kappa_g^2 + \frac{1}{2}\mathfrak{T}^{3(1-\alpha)}\tau_g(\kappa_n^2+\tau_g^2)) \\ &+i(\mathcal{D}^2_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\kappa_n) - \mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{2(1-\alpha)}\kappa_g\tau_g) - \mathfrak{T}^{1-\alpha}\kappa_g\mathcal{D}_{\alpha;\mathcal{T}}(\mathfrak{T}^{1-\alpha}\tau_g) \\ &-\mathfrak{T}^{3(1-\alpha)}\kappa_g^2\kappa_n + \frac{1}{2}\mathfrak{T}^{3(1-\alpha)}\kappa_n(\kappa_n^2+{\tau_g}^2))\varsigma_3^{\alpha} \end{aligned} \tag{62}$$

From Eqs.(61) and (62), the CNLSE is obtained:

$$\mathcal{D}_{\alpha;t}\wp_3^{\alpha} = i\mathcal{D}^2_{\alpha;\mathcal{T}}\wp_3^{\alpha} + \frac{i}{2}(\wp_3^{\alpha *}\wp_3^{\alpha})\wp_3^{\alpha}$$

4 Conclusion

In this study, generalized elastic lines were investigated through intrinsic derivatives with respect to the Darboux frame in Euclidean three-space. The variational approach applied along the intrinsic directional derivatives enabled the derivation of boundary conditions and associated differential equations governing the generalized elastic lines. Moreover, the congruence of the surface $\Sigma^{(\mathfrak{Q})}$ containing the $\mathcal{T}$- and $\mathfrak{Q}$-lines, and the congruence of the surface $\Sigma^{(\mathcal{Z})}$ containing the $\mathcal{T}$- and $\mathcal{Z}$-lines of the Darboux frame in $\mathbb{R}^3$ were explicitly characterized in terms of geodesic curvature, normal curvature, and geodesic torsion. These results provide a deeper understanding of the intrinsic geometry of surfaces and their role in the dynamics of curves.

A further contribution of this work is the introduction of conformal derivatives and conformal magnetic curves associated with the Darboux frame in Euclidean three-space. The Lorentz conformal force equations were established for conformal magnetic curves, thus extending the classical theory of magnetic curves to the conformable setting. In addition, three novel conformal frames and their corresponding Hasimoto-like maps were introduced, which led to conformal differential formulas directly connected with the conformal nonlinear Schrödinger equation (CNLSE). This establishes a new bridge between geometric curve flows and soliton dynamics in the conformal framework.

The obtained results not only enrich the classical theory of curves and surfaces but also reveal potential interdisciplinary applications. In particular, the connection between generalized elastic lines, conformal derivatives, and CNLSE highlights possible applications in optics, waveguide theory, electromagnetism, and quantum mechanics, where the propagation of polarized light, evolution of electric and magnetic

The obtained results not only enrich the classical theory of curves and surfaces but also reveal potential interdisciplinary applications. In particular, the connection between generalized elastic lines, conformal derivatives, and CNLSE highlights possible applications in optics, waveguide theory, electromagnetism, and quantum mechanics, where the propagation of polarized light, evolution of electric and magnetic fields, and soliton-like phenomena can be modeled through the developed framework. Furthermore, the presented formalism suggests new insights into the study of elastic structures, photonic devices, and optomechanical systems, offering promising directions for future research.

In conclusion, this paper provides a unified treatment of generalized elastic lines, congruences of surfaces, conformal derivatives, and magnetic curves for the Darboux frame, while simultaneously establishing novel links with integrable systems. Future work may extend these results to Minkowski space-time

geometries, relativistic particle models, and nonlinear field equations, thereby deepening the interplay between differential geometry and modern physical theories.

Author Contribution Statements

The author of this research paper has directly participated in the planning, execution, or analysis of this study; The author of this paper has read and approved the final version submitted.

Declaration of Competing Interests

The author declares that she has no competing interests.

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This manuscript has no associated data.

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Ethical Standard

This article does not contain any studies with human participants or animals performed by any of the authors.

Declaration of Generative Artificial Intelligence Use

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