International Electronic Journal of Geometry

RESEARCH ARTICLE

e-ISSN: 1307-5624

Vol. 19, No. 2, (2026), 423–441

# Tubular Hypersurfaces with Respect to the Quasi-frame in Four-dimensional Euclidean Space

Başak Yağbasan

Cumali Ekici

Hatice Tozak

Submitted: 29-04-2026 • Accepted: 16-07-2026 • Published Online:25–10–2026

© 2026 Author(s). Licensed under CC BY 4.0.

View Online

**To cite this article:**

Yağbasan B., Ekici C. and Tozak H.: *Tubular Hypersurfaces with Respect to the Quasi-frame in Four-dimensional Euclidean Space*, Int. Electron. J. Geom. 19 (2), 423–441 (2026)

DOI: 10.36890/iejg.1940183

*Discover more articles at:* https://dergipark.org.tr/en/pub/iejg

INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 19 NO. 2 PAGE 423–441 (2026)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1940183
RESEARCH ARTICLE

# Tubular Hypersurfaces with Respect to the Quasi-frame in Four-dimensional Euclidean Space

**Başak Yağbasan, Cumali Ekici* and Hatice Tozak**

(Communicated by Kazım İlarslan)

## ABSTRACT

**This study focuses on the differential geometric analysis of a newly defined class of hypersurfaces within four-dimensional Euclidean space $\mathbb{E}^4$. These hypersurfaces are intrinsically generated as the envelope of a one-parameter family of spheres with constant radius, whose centers trace a space curve with a quasi-frame a generalized orthonormal frame specifically adapted to the curve. Initially, general parametrizations are systematically established for directional canal hypersurfaces, which encompass directional tubular hypersurfaces as a generalized case, and separately for directional tubular hypersurfaces in the four-dimensional Euclidean space $\mathbb{E}^4$. Subsequently, the fundamental geometric properties of directional tubular hypersurface are meticulously investigated. Specifically, the first and second fundamental forms, the unit normal vector field, the shape operator, and including Gaussian curvature and mean curvature are explicitly computed. Finally, this theoretical formulation is exemplified and validated through detailed illustrative examples. These examples include comprehensive symbolic computations and visualizations of the directional tubular hypersurfaces rendered in various projection subspaces, thereby elucidating both the construction methodology and the inherent geometric interpretations of the directional tubular hypersurfaces.**

***Keywords:*** *Hypersurface, Tubular surface, quasi-frame.*

***AMS Subject Classification (2020):*** *Primary: 53A07, Secondary: 14J25; 53A10.*

## 1. Introduction

The theory of curves and surfaces is a significant subject in differential geometry. Hypersurfaces constitute one of the most fundamental structure in differential geometry, representing higher-dimensional analogues of classical surfaces embedded in Euclidean spaces [12,16,20,27,29,35]. Among the wide variety of hypersurfaces, canal hypersurfaces occupy a distinguished position due to their generation from a one-parameter family of spheres. These hypersurfaces naturally generalize classical canal surfaces, defined by one of these researchers, Monge, in three-dimensional Euclidean space and appear in various applications such as computer-aided geometric design, robotics, and physical modeling [17]. Using various moving frames and differential geometric techniques, several authors have investigated different classes of surfaces and their geometric properties in Euclidean spaces. The parametrization of the tubular surface is constructed so that one of its parameters varies along the generating curve described by Blaga [4]. Beyond these fundamental studies, the determination of curvature properties and geometric characterizations of canal and tubular surfaces has attracted considerable attention in both three-dimensional and four-dimensional Euclidean spaces as well as in more general ambient spaces [6, 22, 25, 26, 28, 31, 38]. Despite the extensive research into canal and tubular surfaces, the analysis of Euclidean geometry often relies on Frenet frame structures, which may impose restrictive smoothness conditions on the space curves. In particular, the applicability of Frenet frames

***Received :*** *29-04-2026,* ***Accepted :*** *16-07-2026*
*** Corresponding author***

becomes limited in cases where second-order derivatives are not well-defined. This limitation has motivated the development of alternative frame constructions that provide greater usability in geometric modeling and curvature analysis. For the construction of a three-dimensional offset curve associated with a space curve, the quasi-normal vector is introduced by [8]. The quasi-normal vector is defined at each point of the curve and lies in the plane orthogonal to the tangent vector at that point. At each point on the curve, the quasi-normal vector is defined as a vector lying in the plane normal to the tangent vector [34]. Using this vector-based approach, an alternative frame structure known as the quasi-frame (q-frame), adapted along a space curve, was introduced by Dede et al. [11]. It was shown that this frame can be effectively constructed even in situations where the second derivative of the curve does not exist, thereby overcoming one of the fundamental limitations of the classical Frenet frame [5]. Furthermore, tubular surfaces defined with respect to this frame structure were referred to as directional tubular surfaces, emphasizing the role of the adapted direction field in their geometric construction [10]. Subsequently, a variety of investigations on ruled, rotational, canal and tubular surfaces have been carried out using the quasi-frame in three and four-dimensional Euclidean spaces [3, 13, 21, 36, 39, 40]. In addition to the analytical examination of surfaces generated via the quasi-frame, their visualization also plays a vital role in understanding their geometric structure. Indeed, visual representation provides an effective means of interpreting curvature behavior, local shape variations, and the global configuration of the associated surface. This issue becomes even more significant in higher-dimensional settings, where geometric structures cannot be directly perceived in the usual three-dimensional sense and therefore require suitable projection-based or computational visualization techniques. Gluck, presented a simple algorithm for determining the curvatures of curves in Euclidean n-space using a single formula for all curvatures based on the Gram-Schmidt orthonormalization method, and provided an example to illustrate these concepts in Euclidean 4-space [19]. It was given by Otsuki and Shiohama that a graphical image of a surface exists in $\mathbb{E}^4$ and isometric immersions of the 2-dimensional manifold into the 4-dimensional Euclidean space [30]. Saji, studied the singularities of 2-ruled hypersurfaces in Euclidean 4-space [32]. Studies have also been conducted on tubular hypersurfaces, tubular surface and translation surface in 3 and 4-dimensional space [9, 15, 24, 41]. While the family of hypersurfaces in the Galilean space $\mathbb{G}^4$ was investigated by Yoon and Yüzbaşı [42], parallel hypersurfaces in $\mathbb{E}^4$ were given by Büyükkütük et al [7]. However, despite the extensive literature on canal and tubular surfaces, the systematic investigation of their curvature properties using alternative frame structures—particularly in higher-dimensional Euclidean spaces—remains relatively limited. In this study, we focus on the differential geometric properties of directional tubular hypersurfaces in four-dimensional Euclidean space using quasi-frame, particularly examining their fundamental forms, shape operator and curvature characteristics. The analysis is further extended to include directional tubular hypersurface structures with quasi-frame representations.

## 2. Preliminaries

In this section, the basic notation and fundamental concepts required for the subsequent analysis are briefly presented. Let $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ be three vectors in $\mathbb{E}^4$, where $\mathbf{X} = (x_1, x_2, x_3, x_4)$, $\mathbf{Y} = (y_1, y_2, y_3, y_4)$ and $\mathbf{Z} = (z_1, z_2, z_3, z_4)$. The standart inner product of vectors in $\mathbb{E}^4$, $\mathbf{X}$ and $\mathbf{Y}$ is given as $< \mathbf{X}, \mathbf{Y} >= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$. and the corresponding norm of the vector $\mathbf{X}$ is $||\mathbf{X}|| = \sqrt{< \mathbf{X}, \mathbf{X} >}$. In addition the triple vector product determined by $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ in $\mathbb{E}^4$ may be represented in determinant form as follows

$$\begin{aligned}\mathbf{X} \wedge \mathbf{Y} \wedge \mathbf{Z} = \quad & = (x_2y_3z_4 - x_2y_4z_3 - x_3y_2z_4 + x_3y_4z_2 + x_4y_2z_3 - x_4y_3z_2)\mathbf{E}_1 \\ & +(-x_1y_3z_4 + x_1y_4z_3 + x_3y_1z_4 - x_3y_4z_1 - x_4y_1z_3 + x_4y_3z_1)\mathbf{E}_2 \\ & +(x_1y_2z_4 - x_1y_4z_2 - x_2y_1z_4 + x_2y_4z_1 + x_4y_1z_2 - x_4y_2z_1)\mathbf{E}_3 \\ & +(-x_1y_2z_3 + x_1y_3z_2 + x_2y_1z_3 - x_2y_3z_1 - x_3y_1z_2 + x_3y_2z_1)\mathbf{E}_4\end{aligned}$$

where $\mathbf{E}_1 \wedge \mathbf{E}_2 \wedge \mathbf{E}_3 = \mathbf{E}_4$, $\mathbf{E}_2 \wedge \mathbf{E}_3 \wedge \mathbf{E}_4 = \mathbf{E}_1$, $\mathbf{E}_3 \wedge \mathbf{E}_4 \wedge \mathbf{E}_1 = \mathbf{E}_2$ and $\mathbf{E}_4 \wedge \mathbf{E}_1 \wedge \mathbf{E}_2 = \mathbf{E}_3$ [1, 14, 33, 37]. The geometric role of this vector product becomes more apparent through its fundamental algebraic properties. In particular, its orthogonality, orientation behavior, and linear independence are of significance in relation to the structures examined in the subsequent sections. Let $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ vectors in $\mathbb{E}^4$. When these vectors linearly independent, the vector $\mathbf{X} \wedge \mathbf{Y} \wedge \mathbf{Z} \in \mathbb{E}^4$ is orthogonal to all three vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$. Moreover, the sign of this product changes whenever the positions of any two vectors are exchanged. If the vectors are linearly dependent, then the triple cross product vanishes. Unlike the three-dimensional case, a binary cross product $\mathbf{X} \wedge \mathbf{Y}$ is not defined in $\mathbb{E}^4$, since no unique vector orthogonal to two given vectors can be obtained by an analogous determinant

construction of type $3\times 4$ [1]. Consider a unit-speed space curve $\alpha=\alpha(s)$ in $\mathbb{E}^4$. The corresponding quasi-frame consists of the orthonormal vector fields $\{\mathbf{T},\mathbf{N}_q,\mathbf{B}_q,\mathbf{C}_q\}$, in which $\mathbf{T}$ is the unit tangent vector of the curve, $\mathbf{N}_q$ is the quasi-normal vector field, $\mathbf{B}_q$ and $\mathbf{C}_q$ are the first and second quasi-binormals, respectively. This frame is given by

$$\begin{aligned}\mathbf{T}(s)&=\frac{\alpha'(s)}{\|\alpha'(s)\|}, & \mathbf{N}_q(s)&=\frac{\mathbf{T}(s)\wedge\mathbf{k}_x\wedge\mathbf{k}_y}{\|\mathbf{T}(s)\wedge\mathbf{k}_x\wedge\mathbf{k}_y\|},\\ \mathbf{C}_q(s)&=\frac{\alpha'(s)\wedge\mathbf{N}_q(s)\wedge\alpha'''(s)}{\|\alpha'(s)\wedge\mathbf{N}_q(s)\wedge\alpha'''(s)\|}, & \mathbf{B}_q(s)&=\mathbf{C}_q(s)\wedge\mathbf{T}(s)\wedge\mathbf{N}_s(s)\end{aligned} \tag{2.1}$$

where $\mathbf{k}_x=(1,0,0,0)$ and $\mathbf{k}_y=(0,1,0,0)$ (or $\mathbf{k}_z=(0,0,1,0)$ and $\mathbf{k}_t=(0,0,0,1)$) are the projection vectors. The above definition becomes singular in the case $\mathbf{T}(s)$ contained in the plane spanned by $\mathbf{k}_x$ and $\mathbf{k}_y$. In this situation, the projection vectors may be replaced by an alternative admissible vectors [14, 18]. Suppose now that $\alpha(s)$ is a unit-speed curve parameterized by arc length $t$ in $\mathbb{E}^4$ [18]. Differentiation of (2.1) with respect to $t$, yields the variation formulas of the quasi-frame as follows:

$$\begin{bmatrix}\mathbf{T}'\\ \mathbf{N}_q'\\ \mathbf{B}_q'\\ \mathbf{C}_q'\end{bmatrix}=\begin{bmatrix}0&k_1&k_2&0\\ -k_1&0&k_3&0\\ -k_2&-k_3&0&k_4\\ 0&0&-k_4&0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}_q\\ \mathbf{B}_q\\ \mathbf{C}_q\end{bmatrix}. \tag{2.2}$$

The quasi-curvatures are defined by

$$k_1=\frac{\langle\mathbf{T}',\mathbf{N}_q\rangle}{\|\alpha'\|},\quad k_2=\frac{\langle\mathbf{T}',\mathbf{B}_q\rangle}{\|\alpha'\|},\quad k_3=\frac{\langle\mathbf{N}_q',\mathbf{B}_q\rangle}{\|\alpha'\|}\text{ and }\quad k_4=\frac{\langle\mathbf{B}_q',\mathbf{C}_q\rangle}{\|\alpha'\|}. \tag{2.3}$$

The differential geometry of different types of hypersurfaces $\mathcal{H}$ in 4-dimensional spaces has been a popular topic for geometers, recently [2, 22, 23]. If

$$\begin{aligned}&\varphi:U\subset\mathbb{R}^3\to\mathbb{E}^4,\\ &\varphi(s,u,v)=(\varphi_1(s,u,v),\varphi_2(s,u,v),\varphi_3(s,u,v),\varphi_4(s,u,v))\end{aligned} \tag{2.4}$$

where $\varphi_i, 1\le i\le 4$ denote differentiable real valued functions on the domain $U\subset\mathbb{R}^3$. The image $\varphi(U)\subset\mathbb{E}^4$ constitutes a hypersurface provided that the vector fields $\varphi_s,\varphi_u$ and $\varphi_v$ remain linearly independent throughout the domain These tangent vectors are given by the first order partial derivatives of $\varphi_s$, with respect to the parameters, that is, $\varphi_s=\dfrac{\partial\varphi}{\partial s},\varphi_u=\dfrac{\partial\varphi}{\partial u}$ and $\varphi_v=\dfrac{\partial\varphi}{\partial v}$. Then the corresponding unit normal vector field of the hypersurface $\mathcal{H}$ is obtained as

$$\mathbf{N}(s,u,v)=\frac{\varphi_s\wedge\varphi_u\wedge\varphi_v}{||\varphi_s\wedge\varphi_u\wedge\varphi_v||}. \tag{2.5}$$

The metric structure induced on the hypersurface can be written in the form

$$\mathcal{I}=g_{11}du^2+g_{22}dv^2+g_{33}dw^2+2(g_{12}dudv+g_{13}dudw+g_{23}dvdw) \tag{2.6}$$

with coefficients defined by $g_{ij}=<\varphi_{u_i},\varphi_{u_j}>, 1\le i,j\le 3$ . Accordingly, the matrix expression of the first fundamental form for the hypersurface $\mathcal{H}$ is represented as

$$\mathcal{I}=[g_{ij}]=\begin{bmatrix}g_{11}&g_{12}&g_{13}\\ g_{21}&g_{22}&g_{23}\\ g_{31}&g_{32}&g_{33}\end{bmatrix} \tag{2.7}$$

and also the second fundamental form of the hypersurface is described by

$$\mathcal{II}=h_{11}du^2+h_{22}dv^2+h_{33}dw^2+2(h_{12}dudv+h_{13}dudw+h_{23}dvdw) \tag{2.8}$$

with scalar coefficients $h_{ij}=<\varphi_{u_iu_j},N>, 1\le i,j\le 3$. Accordingly, the associated matrix formulation of $\mathcal{II}$ of the hypersurface $\mathcal{H}$ is given as

$$\mathcal{II}=[h_{ij}]=\begin{bmatrix}h_{11}&h_{12}&h_{13}\\ h_{21}&h_{22}&h_{23}\\ h_{31}&h_{32}&h_{33}\end{bmatrix}. \tag{2.9}$$

Here $\varphi_{u_i} = \dfrac{\partial\varphi(u_1,u_2,u_3)}{\partial u_i}, \varphi_{u_iu_j} = \dfrac{\partial^2\varphi(u_1,u_2,u_3)}{\partial u_j\partial u_i}, 1 \le i,j \le 3$. Also the shape operator matrix of the hypersurface (2.4) is

$$\mathcal{S} = \mathcal{I}^{-1}.\mathcal{II} = [g^{ij}].[h_{ij}] = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \tag{2.10}$$

where $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$. With the aid of (2.5), (2.7) and (2.9), the Gaussian curvature $K$ and mean curvature $H$ of a hypersurface in $\mathbb{E}^4$ are formulated as follows:

$$K = \det(\mathcal{S}) = \frac{\det[h_{ij}]}{\det[g_{ij}]} \quad \text{and} \quad H = \frac{1}{3}tr(\mathcal{S}) \tag{2.11}$$

respectively [23]. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be a center curve in the four-dimensional Euclidean space $\mathbb{E}^4$ with the arc-length parameter $s$ and the unit tangent vector $\mathbf{E}_0$. Let $\{\mathbf{E}_1(s), \mathbf{E}_2(s), \mathbf{E}_3(s)\}$ be an orthonormal basis of the directrix space (i.e., it is a 3-dimensional subspace perpendicular to the tangent vector) along this curve. Then [22], a tubular hypersurface with the center curve $\alpha$ is parametrized by

$$\begin{aligned} &\varphi : I \times \mathbb{R}^2 \to \mathbb{E}^4 \\ &\varphi(s,u,v) = \alpha(s) + r(\cos u \sin v\mathbf{E}_1(s) + \sin u \sin v\mathbf{E}_2(s) + \cos v\mathbf{E}_3(s)). \end{aligned} \tag{2.12}$$

## 3. The Tubular Hypersurfaces Generated by the Quasi-Vectors

The general equation of a canal hypersurface with quasi-frame called a directional canal hypersurface in 4-dimensional space $\mathbb{E}^4$, where $r(s)$ is a function of radius, is defined as

$$\varphi(s,u,v) = \alpha(s)+r(s)(\cos u \cos v\mathbf{N}_q(s)+ \sin u \cos v\mathbf{B}_q(s)+ \sin v\mathbf{C}_q(s)). \tag{3.1}$$

As a special case of the directional canal hypersurface introduced for the first time in this study and given in Eq. (3.1), the general equation of the tubular hypersurface with quasi-frame in four-dimensional Euclidean space, referred to as the directional tubular hypersurface, is presented by

$$\varphi(s,u,v) = \alpha(s)+r(\cos u \cos v\mathbf{N}_q(s)+ \sin u \cos v\mathbf{B}_q(s)+ \sin v\mathbf{C}_q(s)). \tag{3.2}$$

The following theorem provides the explicit form of the unit normal vector field of the directional tubular hypersurface constructed along a unit-speed curve in $\mathbb{E}^4$ within the quasi-frame.

**Theorem 3.1.** *Let $\alpha : I \to \mathbb{E}^4$ be a unit-speed curve with the quasi-frame $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q, \mathbf{C}_q\}$. Let $\varphi$ be the directional tubular hypersurface in $\mathbb{E}^4$ of radius $r \in \mathbb{E}$ having $\alpha$ as its center curve. Then, the corresponding unit normal vector field of $\varphi$ is given by*

$$\mathbf{N} = (0, \cos u \cos v, \sin u \cos v, \sin v). \tag{3.3}$$

*Proof.* Taking partial derivatives of the parametrization with respect to $s$, $u$, and $v$, we obtain the following first-order partial derivatives as

$$\begin{aligned} \varphi_s = \;& (1 - rk_1 \cos u \sin v - rk_2 \sin u \sin v)\mathbf{T}(s) - rk_3 \sin u \sin v\mathbf{N}_q(s) \\ & +(rk_3 \cos u \sin v - rk_4 \cos v)\mathbf{B}_q(s) + rk_4 \sin u \sin v\mathbf{C}_q(s) \\ \varphi_u = \;& r \sin v(-\sin u\mathbf{N}_q(s) + \cos u\mathbf{B}_q(s)) \\ \varphi_v = \;& r(\cos u \cos v\mathbf{N}_q(s) + \sin u \cos v\mathbf{B}_q(s) - \sin v\mathbf{C}_q(s)). \end{aligned} \tag{3.4}$$

Using these first-order partial derivatives, the second-order partial derivatives are derived as follows

$$\begin{aligned}
\varphi_{ss} = \; & (rk_2k_4\cos v + rk_3\sin v(k_1\sin u - k_2\cos u) - rk_1'\cos u\sin v - rk_2'\sin u\sin v)\mathbf{T}(s) \\
& +(k_1 - r\cos u\sin v(k_1^2+k_3^2) - r\sin u\sin v(k_3'+k_1k_2) + rk_3k_4\cos v)\mathbf{N}_q(s) \\
& +(k_2 + r\cos u\sin v(k_3' - k_1k_2) - r\sin u\sin v(k_4^2+k_2^2+k_3^2) - rk_4'\cos v)\mathbf{B}_q(s) \\
& +(r\sin v(k_3k_4\cos u + k_4'\sin u) - rk_4^2\cos v)\mathbf{C}_q(s) \\
\varphi_{uu} = \; & -r\sin v(\cos u\mathbf{N}_q(s) + \sin u\mathbf{B}_q(s)) \\
\varphi_{vv} = \; & -r(\cos u\sin v\mathbf{N}_q(s) + \sin u\sin v\mathbf{B}_q(s) + \cos v\mathbf{C}_q(s))
\end{aligned}$$

and

$$\begin{aligned}
\varphi_{su} = \varphi_{us} = \; & -r\sin v((k_2\cos u - k_1\sin u)\mathbf{T}(s) + k_3\cos u\mathbf{N}_q(s) + k_3\sin u\mathbf{B}_q(s) \\
& -k_4\cos u\mathbf{C}_q(s)) \\
\varphi_{sv} = \varphi_{vs} = \; & -r((k_1\cos u\cos v + k_2\sin u\cos v)\mathbf{T}(s) + k_3\sin u\cos v\mathbf{N}_q(s) \\
& -(k_4\sin v + k_3\cos u\cos v)\mathbf{B}_q(s) - k_4\sin u\cos v\mathbf{C}_q(s)) \\
\varphi_{uv} = \varphi_{vu} = \; & -r\cos v(\sin u\mathbf{N}_q(s) - \cos u\mathbf{B}_q(s))
\end{aligned}$$

Substituting the partial derivatives into the expression for the unit normal vector field as defined in Equation (2.5), we compute:

$$\varphi_s \times \varphi_u \times \varphi_v = \frac{(0,\; r^2\cos^2 v\cos u,\; r^2\cos^2 v\sin u,\; r^2\cos v\sin v)}{(rk_1\cos u\cos v + rk_2\sin u\cos v - 1)^{-1}}.$$

Thus, the norm is

$$||\varphi_s \times \varphi_u \times \varphi_v|| = r^2\cos v(rk_1\cos u\cos v + rk_2\sin u\cos v - 1).$$

Consequently, the unit normal vector field $N(s,u,v)$ of the directional tubular hypersurface is given by

$$\mathbf{N} = (0, \cos u\cos v, \sin u\cos v, \sin v). \tag{3.5}$$

The next theorem presents the Gaussian curvature and mean curvature of the directional tubular hypersurface, obtained from the shape operator matrix derived above. □

**Theorem 3.2.** *Let $\varphi$ be the directional tubular hypersurface in $\mathbb{E}^4$ associated with the quasi-frame construction. Then, the Gaussian curvature $K$ and the mean curvature $H$ of the directional tubular hypersurface $\varphi$, computed from the previously obtained shape operator matrix, are, respectively, given by*

$$\begin{aligned}
K = \; & \frac{-g^{11}}{r^2}(\cos v(rk_1\cos u\cos v(k_1\cos u + 2k_2\sin u) + k_2\sin u(rk_2\cos v\sin u - 1) - k_1\cos u)) \\
H = \; & \frac{-g^{11}}{3r}(5r\cos v(k_2\sin u - k_1\cos u) + 3r^2\cos^2 u\cos^2 v(k_1^2 - k_2^2) \\
& +3r^2k_2\cos^2 v(2k_1\cos u\sin u + k_2) + 2),
\end{aligned} \tag{3.6}$$

*where $g^{11}$ is the $(1,1)$-component of the inverse of the first fundamental form matrix.*

*Proof.* From Eq. (3.4), the coefficients of the first fundamental form are computed as follows:

$$\begin{aligned}
g_{11} = \; & 1 - 2r\cos u\sin v(k_1 + rk_3k_4\cos v) - 2rk_2\sin u\sin v + r^2\sin^2 vk_3^2 \\
& +r^2k_4^2(\cos^2 u\cos^2 v + \sin^2 u) + r^2\sin^2 v(k_1\cos u + k_2\sin v)^2
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
g_{12} &= r^2\sin v(k_3\sin v - k_4\cos u\cos v) & g_{23} &= 0 \\
g_{13} &= -r^2k_4\sin u & g_{31} &= -r^2k_4\sin u \\
g_{21} &= r^2\sin v(k_3\sin v - k_4\cos u\cos v) & g_{32} &= 0 \\
g_{22} &= r^2\sin^2 v & g_{33} &= r^2.
\end{aligned} \tag{3.8}$$

The matrix of the first fundamental form is presented below

$$\mathcal{I} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}. \tag{3.9}$$

To compute the inverse of the first fundamental form matrix, we apply algebraic simplifications based on the structure of the coefficients. The elements $g_{ij}$ of the inverse matrix $\mathcal{I}^{-1}$ are explicitly derived as

$$\mathcal{I}^{-1} = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} \tag{3.10}$$

where

$$\begin{aligned}
g^{11} &= \frac{1}{(1\text{-}rk_1\cos u\cos v)^2+r^2k_2^2\cos^2 v\sin^2 u\text{-}2rk_2\sin u\cos v(1+rk_1\cos u\cos v)} \\
g^{12}=g^{21} &= \frac{g^{11}(k_4\cos u\sin v-k_3\cos v)}{\cos v} \\
g^{13}=g^{31} &= g^{11}(-k_4\sin u) \\
g^{22} &= \frac{(g^{11})^{-1}+r^2\cos^2 v(k_3^2\text{-}k_4^2\cos^2 u)\text{-}2r^2k_3k_4\cos u\cos v\sin v+r^2k_4^2\cos^2 u}{r^2\cos^2 v(g^{11})^{-1}} \\
g^{23}=g^{32} &= \frac{k_4\sin u(k_3\cos v-k_4\cos u\sin v)(g^{11})^{-1}}{\cos v} \\
g^{33} &= \frac{g^{11}}{r^4\cos v}(r^2\cos^2 v((g^{11})^{-1}+r^2k_4^2(1-\cos^2 u\cos^2 v)+r^2k_3\cos v(k_3\cos v-2k_4\cos u\sin v)) \\
&\quad -r^4\cos^2 v(k_4\cos u\sin v-k_3\cos v)^2).
\end{aligned} \tag{3.11}$$

Similarly, the coefficients of the second fundamental form are obtained using the second-order partial derivatives and $\mathbf{N}(s,u,v)$ the unit normal vector

$$\begin{aligned}
h_{11} &= \cos u\cos v(k_1\text{+}2rk_3k_4\cos u\cos v)+r\cos^2 u\cos^2 v(k_4^2-k_1^2\text{+}k_2^2) \\
&\quad -rk_4^2\text{+}k_2\sin u\cos v(1-2rk_1\cos u\cos v)-r\cos^2 v(k_2^2\text{+}k_3^2)
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
h_{12} &= -r\cos v(k_3\cos v-k_4\cos u\sin v) & h_{23} &= 0 \\
h_{13} &= -r^2k_4\sin u & h_{31} &= -rk_4\sin u \\
h_{21} &= -r\cos v(k_3\cos v-k_4\cos u\sin v) & h_{32} &= 0 \\
h_{22} &= -r\cos^2 v & h_{33} &= -r.
\end{aligned} \tag{3.13}$$

The second fundamental form matrix is

$$\mathcal{II} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}. \tag{3.14}$$

The shape operator $\mathcal{S}$ corresponding to the tubular hypersurface takes the following form

$$\mathcal{S} = \begin{bmatrix} -g^{11}\cos v(rk_1\cos u\cos v(k_1\cos u+2k_2\sin u)-\rho+rk_2^2\cos v\sin^2 u) & 0 & 0 \\ -\dfrac{g^{11}}{r}(k_3(1-r\rho\cos v)+k_4\cos u\sin v(r\rho-\cos^{-1}v)) & -\dfrac{1}{r} & 0 \\ \dfrac{g^{11}}{r}k_4\sin u(r\rho\cos v-1) & 0 & -\dfrac{1}{r} \end{bmatrix}$$

where $\rho = k_1 \cos u + k_2 \sin u$. In order to further characterize the hypersurface from a geometric perspective, the Gaussian curvature $K$ and the mean curvature $H$ are

$$\begin{aligned} K = \; & \frac{-g^{11}}{r^2}(\cos v(rk_1 \cos u \cos v(k_1 \cos u + 2k_2 \sin u) - k_1 \cos u + rk_2^2 \cos v \sin^2 u - k_2 \sin u)) \\ H = \; & \frac{-g^{11}}{3r}(5r \cos v(k_2 \sin u - k_1 \cos u) + 3r^2 \cos^2 u \cos^2 v(k_1^2 - k_2^2) \\ & + 3r^2 k_2 \cos^2 v(2k_1 \cos u \sin u + k_2) + 2), \end{aligned} \tag{3.15}$$

respectively. □

**Theorem 3.3.** *Let $\varphi$ be the directional tubular hypersurface in $\mathbb{E}^4$ associated with the quasi-frame. Then, the eigenvalues of the corresponding shape operator, namely the principal curvatures of $\varphi$, are obtained as*

$$\begin{aligned} \lambda_1 = \; & g^{11} \cos v\big(r \cos v(k_1 \cos u + k_2 \sin u)^2 - k_1 \cos v - k_2 \sin u\big), \\ \lambda_2 = \; & -\frac{1}{r}, \\ \lambda_3 = \; & -\frac{1}{r}, \end{aligned} \tag{3.16}$$

*where $g^{11}$ is the $(1,1)$-component of the inverse of the first fundamental form matrix.*

*Proof.* It follows directly from the computations carried out above that the principal curvatures are determined as the eigenvalues of the shape operator matrix of the directional tubular hypersurface $\varphi$. Evaluating the corresponding characteristic equation yields the expressions given in Eq. (3.16). □

**Corollary 3.1.** *In addition to the results obtained above, it is observed that the hypersurface is flat when $v = \frac{\pi}{2}$ is taken for the Gaussian curvature given by Equation (3.15).*

**Example 3.1.** Let us consider a specific example to illustrate the structure andgeometric analysis of a directional tubular hypersurface in 4-dimensional Euclidean space $\mathbb{E}^4$ with quasi frame. Let

$$\alpha(s) = (4 + \frac{1}{2\sqrt{2}} \sin 2s, \frac{1}{2\sqrt{2}} \cos 2s, \frac{1}{3\sqrt{2}} \sin 3s, 5 - \frac{1}{3\sqrt{2}} \cos 3s) \tag{3.17}$$

be a space curve, which will use as centre curve with quasi-frame of tubular hypersurface in $\mathbb{E}^4$. It can be easily showed that $\|\alpha'(s)\| = 1$, and hence the curve $\alpha(s)$ is parameterized by the arc length. Accordingly, by employing the relations given in Eq. in (2.1), the associated quasi-vectors of $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q, \mathbf{C}_q\}$ can be constructed as

$$\begin{aligned} \mathbf{T}(s) = \; & (\frac{1}{\sqrt{2}} \cos 2s, -\frac{1}{\sqrt{2}} \sin 2s, \frac{1}{\sqrt{2}} \cos 3s, \frac{1}{\sqrt{2}} \sin 3s) \\ \mathbf{N}_q(s) = \; & (0, 0, \sin 3s, -\cos 3s) \\ \mathbf{B}_q(s) = \; & (-\frac{1}{\sqrt{2}} \cos 2s, \frac{1}{\sqrt{2}} \sin 2s, \frac{1}{\sqrt{2}} \cos 3s, \frac{1}{\sqrt{2}} \sin 3s) \\ \mathbf{C}_q(s) = \; & (\sin 2s, \cos 2s, 0, 0). \end{aligned} \tag{3.18}$$

Based on this quasi-frame, the corresponding quasi-curvatures are computed using Eq. (2.3) and are given by

$$k_1(s) = -\frac{3}{\sqrt{2}}, \quad k_2(s) = 0, \quad k_3(s) = \frac{3}{\sqrt{2}} \quad \text{and} \quad k_4(s) = \sqrt{2}. \tag{3.19}$$

Replacing the relevant terms in equation (3.2) with the expressions given in (3.17) and (3.18) yields the following parametrization of the directional tubular hypersurface is

parametrized as

$$\begin{aligned}\varphi(s,u,v) = \quad & (4+\frac{1}{2\sqrt{2}}\sin 2s+\frac{r}{\sqrt{2}}\cos v\sin u\cos 2s+r\sin v\sin 2s,\\ & \frac{1}{2\sqrt{2}}\cos 2s+\frac{r}{\sqrt{2}}\cos v\sin u\sin 2s+r\sin v\cos 2s,\\ & \frac{1}{2\sqrt{2}}\sin 3s+r\cos u\cos v\sin 3s+\frac{r}{\sqrt{2}}\sin u\cos v\cos 3s,\\ & 5-\frac{1}{2\sqrt{2}}\cos 3s-r\cos u\cos v\cos 3s+\frac{r}{\sqrt{2}}\sin u\cos v\sin 3s).\end{aligned}$$

For the specific case where $r=\frac{3}{2}$, the parametrization of tubular hypersurface is

$$\begin{aligned}\varphi(s,u,v) = \quad & (4+\frac{1}{2\sqrt{2}}\sin 2s+\frac{3}{2}(\frac{1}{\sqrt{2}}\cos v\sin u\cos 2s+\sin v\sin 2s),\\ & \frac{1}{2\sqrt{2}}\cos 2s+\frac{3}{2}(\frac{1}{\sqrt{2}}\cos v\sin u\sin 2s+\sin v\cos 2s),\\ & \frac{1}{2\sqrt{2}}\sin 3s+\frac{3}{2}\cos v(\cos u\sin 3s+\frac{1}{\sqrt{2}}\sin u\cos 3s),\\ & 5-\frac{1}{2\sqrt{2}}\cos 3s-\frac{3}{2}\cos v(\cos u\cos 3s-\frac{1}{\sqrt{2}}\sin u\sin 3s)).\end{aligned} \tag{3.20}$$

From this parametrization, partial derivatives with respect to $s$, $u$, and $v$ are computed, followed by the second-order derivatives. These are used to derive the unit normal vector field and the coefficients of the fundamental forms

$$\begin{aligned}\varphi_s = \quad & (\frac{1}{\sqrt{2}}\cos 2s+3\sqrt{2}\sin u\cos v\cos s\sin s+3\sin v\cos 2s,\\ & -\frac{1}{\sqrt{2}}\sin 2s+\frac{3}{\sqrt{2}}\sin u\cos v\cos 2s-6\sin v\cos s\sin s,\\ & \frac{1}{\sqrt{2}}\cos 3s+\frac{9}{2}\cos u\cos v\cos 3s-3\sin u\cos v\sin s(\frac{4\cos^2 s-3}{2\sqrt{2}}+2\sqrt{2}\cos^2 s),\\ & \frac{1}{\sqrt{2}}\sin 3s+\frac{9}{2}\cos u\cos v\sin 3s+3\sin u\cos v\cos s(\frac{4\cos^2 s-1}{2\sqrt{2}}-2\sqrt{2}\sin^2 s))\\ \varphi_u = \quad & (-\frac{3}{2\sqrt{2}}\cos u\cos v(2\cos^2 s-1),\frac{3}{2\sqrt{2}}\cos u\cos v\cos s\sin s,\\ & \frac{3}{2}\cos v(\frac{1}{\sqrt{2}}\cos u\cos s(4\cos^2 s-3)-\sin u\sin 3s),\\ & \frac{3}{2}\cos v(\frac{1}{\sqrt{2}}\cos u\sin s(4\cos^2 s-1)+\sin u\cos 3s))\end{aligned}$$

and

$$\begin{aligned}\varphi_v = \quad & (\frac{3}{2\sqrt{2}}\sin u\sin v(2\cos^2 s-1)+3\cos v\sin s\cos s,\\ & \frac{3}{2}\cos v(2\cos^2 s-1)-\frac{3}{\sqrt{2}}\sin u\sin v\cos s\sin s,\\ & -\frac{3}{2}\cos u\sin v\sin 3s-\frac{3}{2\sqrt{2}}\sin u\sin v\cos t(4\cos^2 s-3),\\ & \frac{3}{2}\sin v(\cos u\cos 3s-\frac{1}{\sqrt{2}}\sin u\sin s(4\cos^2 s-1))).\end{aligned}$$

Then differentiating $\varphi_s$, $\varphi_u$ and $\varphi_v$ with respect to $s$, $u$ and $v$ gives

$$\begin{aligned}
\varphi_{ss} = \quad & (3\sqrt{2}\sin u\cos v\cos^2 s - \sqrt{2}\sin 2s - 3\sqrt{2}\sin u\cos v\sin^2 s - 12\sin v\sin s\cos s,\\
& 6\sin v\sin^2 s - \sqrt{2}\cos 2s - 6\sqrt{2}\sin u\cos v\cos s\sin s - 6\sin v\cos^2 s,\\
& -\frac{3}{\sqrt{2}}\sin 3s - \frac{27}{2}\cos u\cos v\sin 3s - \frac{3}{2\sqrt{2}}\sin u\cos v\cos s(4\cos^2 s - 3)\\
& +18\sqrt{2}\sin u\cos v\sin^2 s\cos s - 6\sqrt{2}\sin u\cos v\cos^3 s,\\
& \frac{3}{\sqrt{2}}\cos 3s + \frac{27}{2}\cos u\cos v\cos 3s + 6\sqrt{2}\sin u\cos v\sin s(\sin s - 3\cos^2 s)\\
& -\frac{3}{2\sqrt{2}}\sin u\cos v\sin s(4\cos^2 s - 1))\\
\varphi_{su} = \varphi_{us} = \quad & (3\sqrt{2}\cos u\cos v\cos s\sin s, -\frac{3}{\sqrt{2}}\cos u\cos v\sin^2 s + \frac{3}{\sqrt{2}}\cos u\cos v\cos^2 s,\\
& -\frac{9}{2}\sin u\cos v\cos 3s - 3\cos u\cos v\sin s(\frac{4\cos^2 s - 3}{2\sqrt{2}} + 2\sqrt{2}\cos^2 s),\\
& -\frac{9}{2}\sin u\cos v\sin 3s + 3\cos u\cos v\cos s(\frac{4\cos^2 s - 1}{2\sqrt{2}} - 2\sqrt{2}\sin^2 s))
\end{aligned}$$

and

$$\begin{aligned}
\varphi_{sv} = \varphi_{vs} = \quad & (-3\sqrt{2}\sin u\sin v\cos s\sin s + 3\cos v\cos^2 t - 3\cos v\sin^2 s,\\
& \frac{3}{\sqrt{2}}\sin u\sin v\sin^2 s - \frac{3}{\sqrt{2}}\sin u\sin v\cos^2 s - 6\cos v\sin s\cos s,\\
& -\frac{9}{2}\cos u\sin v\cos 3s + 3\sin u\sin v\sin s(\frac{4\cos^2 s - 3}{2\sqrt{2}} + 2\sqrt{2}v\cos^2 s),\\
& -\frac{9}{2}\cos u\sin v\sin 3s + 3\cos s\sin u\sin v(2\sqrt{2}\sin^2 s - \frac{4\cos^2 s - 1}{2\sqrt{2}}))
\end{aligned}$$

and

$$\begin{aligned}
\varphi_{uu} = \quad & (\frac{3}{2\sqrt{2}}\sin u\cos v(2\cos^2 s - 1), -\frac{3}{\sqrt{2}}\sin u\cos v\cos s\sin s,\\
& -\frac{3}{2}\cos u\cos v\sin 3s - \frac{3}{2\sqrt{2}}\sin u\cos v\cos s(4\cos^2 s - 3),\\
& \frac{3}{2}\cos u\cos v\cos 3s - \frac{3}{2\sqrt{2}}\sin u\cos v\sin s(4\cos^2 s - 1))\\
\varphi_{uv} = \varphi_{vu} = \quad & (\frac{3}{2\sqrt{2}}\cos u\sin v(2\cos^2 s - 1), -\frac{3}{\sqrt{2}}\cos u\sin v\cos s\sin s,\\
& \frac{3}{2}\sin u\sin v\sin 3s - \frac{3}{2\sqrt{2}}\cos u\sin v\cos s(4\cos^2 s - 3),\\
& -\frac{3}{2}\sin u\sin v\cos 3s - \frac{3}{2\sqrt{2}}\cos u\sin v\sin s(4\cos^2 t - 1))
\end{aligned}$$

and,

$$\begin{aligned}
\varphi_{vv} = \quad & (\frac{3}{2\sqrt{2}}\sin u\cos v(2\cos^2 s - 1) - 3\sin v\sin s\cos s,\\
& -\frac{3}{\sqrt{2}}\sin u\cos v\cos s\sin s - \frac{3}{2}\sin v(2\cos^2 s - 1),\\
& -\frac{3}{2}\cos v(\cos u\sin 3s - \frac{1}{\sqrt{2}}\sin u\cos s(4\cos^2 s - 3)),\\
& \frac{3}{2}\cos v(\cos u\cos 3s - \frac{1}{\sqrt{2}}\sin u\sin s(4\cos^2 s - 1))),
\end{aligned}$$

respectively. At a representative point, for instance, $P = \varphi(\frac{\pi}{2}, u, \pi)$, the unit normal vector field, fundamental form matrices, shape operator, Gaussian, mean and principal curvatures, of tubular hypersurface are calculated. Then, for tubular hypersurfaces the unit normal vector field can be calculated as

$$\mathbf{N}_\varphi = \frac{1}{\delta}(\sin u(-2\sqrt{2} + 9\cos u), 0, \cos u(4 - 9\sqrt{2}\cos u), \sin u(2\sqrt{2} - 9\cos u))$$

where $\delta = 18\cos u(\cos u - 8\sqrt{2}) + 16$. In this case, the matrix of first fundamental form coefficients of the tubular hypersurface in 4-dimensional Euclidean space

$$\mathcal{I} = \begin{bmatrix} \frac{125}{8} - \frac{9}{\sqrt{2}}\cos u + \frac{45}{8}\cos^2 u & \frac{27}{4\sqrt{2}} & \frac{9}{2\sqrt{2}}\sin u \\ \frac{27}{4\sqrt{2}} & \frac{9}{4} & 0 \\ \frac{9}{2\sqrt{2}}\sin u & 0 & \frac{9}{4} \end{bmatrix}. \tag{3.21}$$

Then the inverse of the first fundamental form coefficients matrix from equation (3.21) is

$$\mathcal{I}^{-1} = \frac{1}{\Delta}\begin{bmatrix} 8 & -12\sqrt{2} & -8\sqrt{2}\sin u \\ -12\sqrt{2} & \frac{4(89 - 36\sqrt{2}\cos u + 81\cos^2 u)}{9} & 24\sin u \\ -8\sqrt{2}\sin u & 24\sin u & \frac{4(44 - 36\sqrt{2}\cos u + 45\cos^2 u)}{9} \end{bmatrix} \tag{3.22}$$

where $\Delta = 9\cos u(9\cos u - 4\sqrt{2}) + 8$. The second fundamental from coefficients matrix is

$$\mathcal{II} = \frac{3}{2.\delta}\begin{bmatrix} \frac{\cos u(125\sqrt{2} + 45\cos^2 u - 56\cos u) - 52}{2} & 3(9\cos u - 2\sqrt{2}) & \sin u(9\cos u - 2\sqrt{2}) \\ 3(9\cos u - 2\sqrt{2}) & 9\sqrt{2}\cos u - 4 & 0 \\ \sin u(9\cos u - 2\sqrt{2}) & 0 & 9\sqrt{2}\cos u - 4 \end{bmatrix}. \tag{3.23}$$

The shape operator matrix of the tubular hypersurface $\varphi(s,u,v)$ in $\mathbb{E}^4$ space is the product of the matrices (3.22) and (3.23), hence the shape operator matrix is

$$\mathcal{S} = \frac{1}{\Delta.\delta}\begin{bmatrix} 6\cos u(8\sqrt{2} + 9\cos u(9\sqrt{2}\cos u - 8)) & 0 & 0 \\ -4(8\sqrt{2} + 9\cos u(9\sqrt{2}\cos u - 8) & \frac{2.\mathcal{A}}{3} & 0 \\ \frac{8\sin u(8\sqrt{2} + 9\cos u(9\sqrt{2}\cos u - 8))}{3} & 0 & \frac{2.\mathcal{A}}{3} \end{bmatrix},$$

where $\mathcal{A} = 216\sqrt{2}\cos u - 32 - 972\cos^2 u + 729\sqrt{2}\cos^3 u$. By using equations (2.11) the Gaussian and mean curvatures of the tubular hypersurface are explicitly obtained as

$$\begin{aligned} K \quad &= 8\cos u\Big[\frac{\sqrt{2}\cos^2 u(43046721\cos^6 u + 119042784\cos^4 u + 29393280\cos^2 u + 1161216)}{3.\Delta.\delta.\mathcal{B}} \\ &+ \frac{4096\sqrt{2} - \cos u(10450944\cos^2 u + 105815808\cos^4 u + 153055008\cos^6 u + 147456)}{3.\Delta.\delta.\mathcal{B}}\Big] \end{aligned}$$

and

$$H = \frac{2(504\sqrt{2}\cos u + 2187\sqrt{2}\cos^3 u - 2592\cos^2 u - 64)}{9.\Delta.\delta}$$

where $\mathcal{B} = \cos^2 u(531441\cos^4 u + 787320\cos^2 u + 77760) - \sqrt{2}\cos u(708588\cos^4 u + 233280\cos u + 6912) + 512$, respectively. It is easily seen that if $u = \frac{\pi}{2}$ is taken, the value of the Gaussian curvature becomes zero, in which case the hypersurface is flat. From these curvature expressions, the principal curvatures of the directional tubular hypersurface are determined as follows:

$$\begin{aligned} \lambda_1 &= \frac{6\cos u(8\sqrt{2} + 81\sqrt{2}\cos^2 u - 72\cos u)}{\Delta.\delta} \\ \lambda_2 &= \frac{2(216\sqrt{2}\cos u - 972\cos^2 u + 729\sqrt{2}\cos^3 u - 32)}{3.\Delta.\delta} \\ \lambda_3 &= \frac{2(216\sqrt{2}\cos u - 972\cos^2 u + 729\sqrt{2}\cos^3 u - 32)}{3.\Delta.\delta}. \end{aligned}$$

To visualize the geometric structure more clearly, we first consider the special case $v = 0$ in the general parametrization given by Eq. (3.20). Substituting this condition into the hypersurface representation yields the following parametric form:

$$\begin{aligned}\varphi(s,u,0) = \; & (4+\frac{1}{2\sqrt{2}}\sin 2s+\frac{3}{2\sqrt{2}}\sin u\cos 2s, \frac{1}{2\sqrt{2}}\cos 2s+\frac{3}{2\sqrt{2}}\sin u\sin 2s,\\ & \frac{1}{2\sqrt{2}}\sin 3s+\frac{3}{2}\cos u\sin 3s+\frac{3}{2\sqrt{2}}\sin u\cos 3s,\\ & 5-\frac{1}{2\sqrt{2}}\cos 3s-\frac{3}{2}\cos u\cos 3s+\frac{3}{2\sqrt{2}}\sin u\sin 3s).\end{aligned} \tag{3.24}$$

More precisely, the projection of the hypersurface onto the $xyz$-subspace is given by Eq. (3.24), while the corresponding projections onto the $xyt-$, $xzt-$, and $yzt-$subspaces are presented in Eqs. (3.25)-(3.28)

$$\begin{aligned}\varphi(s,u) = \; & (4+\frac{1}{2\sqrt{2}}\sin 2s+\frac{3}{2\sqrt{2}}\sin u\cos 2s, \frac{1}{2\sqrt{2}}\cos 2s+\frac{3}{2\sqrt{2}}\sin u\sin 2s,\\ & \frac{1}{2\sqrt{2}}\sin 3s+\frac{3}{2}\cos u\sin 3s+\frac{3}{2\sqrt{2}}\sin u\cos 3s),\end{aligned} \tag{3.25}$$

$$\begin{aligned}\varphi(s,u) = \; & (4+\frac{1}{2\sqrt{2}}\sin 2s+\frac{3}{2\sqrt{2}}\sin u\cos 2s, \frac{1}{2\sqrt{2}}\cos 2s+\frac{3}{2\sqrt{2}}\sin u\sin 2s,\\ & 5-\frac{1}{2\sqrt{2}}\cos 3s-\frac{3}{2}\cos u\cos 3s+\frac{3}{2\sqrt{2}}\sin u\sin 3s),\end{aligned} \tag{3.26}$$

$$\begin{aligned}\varphi(s,u) = \; & (4+\frac{1}{2\sqrt{2}}\sin 2s+\frac{3}{2\sqrt{2}}\sin u\cos 2s, \frac{1}{2\sqrt{2}}\sin 3s+\frac{3}{2}\cos u\sin 3s+\frac{3}{2\sqrt{2}}\sin u\cos 3s,\\ & 5-\frac{1}{2\sqrt{2}}\cos 3s-\frac{3}{2}\cos u\cos 3s+\frac{3}{2\sqrt{2}}\sin u\sin 3s),\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}\varphi(s,u) = \; & (\frac{1}{2\sqrt{2}}\cos 2s+\frac{3}{2\sqrt{2}}\sin u\sin 2s, \frac{1}{2\sqrt{2}}\sin 3s+\frac{3}{2}\cos u\sin 3s+\frac{3}{2\sqrt{2}}\sin u\cos 3s),\\ & 5-\frac{1}{2\sqrt{2}}\cos 3s-\frac{3}{2}\cos u\cos 3s+\frac{3}{2\sqrt{2}}\sin u\sin 3s),\end{aligned} \tag{3.28}$$

respectively. The graphical representations of the hypersurface defined by the parametrization $\varphi(s,u,0)$ given in Eq. (3.24) are presented in Figure 1 in the $xyz-$, $xyt-$, $xzt-$ and $yzt-$projection spaces, together with the associated center curve. These projections provide a clearer geometric interpretation of the hypersurface by revealing how its local tubular structure appears in different three-dimensional subspaces.

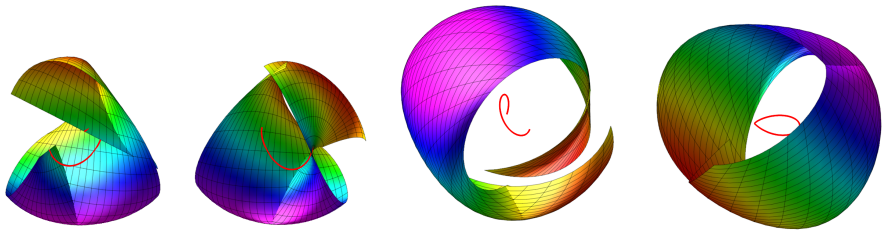


**Figure 1.** Projections of the directional tubular hypersurface $\varphi(s,u,0)$ onto the $xyz-$, $xyt-$, $xzt-$, and $yzt$-spaces.

Next, another family of sections of the hypersurface is investigated by fixing $u = 0$ and considering the parametrization $\varphi(s,0,v)$. Substituting this condition into the general hypersurface equation produces the following parametrization:

$$\begin{aligned}\varphi(s,0,v) = \; & (4+\frac{1}{2\sqrt{2}}\sin 2s+\frac{3}{2}\sin v\sin 2s, \frac{1}{2\sqrt{2}}\cos 2s+\frac{3}{2}\sin v\cos 2s,\\ & \frac{1}{2\sqrt{2}}\sin 3s+\frac{3}{2}\cos v\sin 3s, 5-\frac{1}{2\sqrt{2}}\cos 3s-\frac{3}{2}\cos v\cos 3s).\end{aligned} \tag{3.29}$$

The induced projections of $\varphi(s,0,v)$ onto the $xyz, xyt-, xzt-$ and $yzt-$subspaces are given in Eqs. (3.30)-(3.33)

$$\begin{aligned}\varphi(s,v) = \quad & (4+\frac{1}{2\sqrt{2}}\sin 2s+\frac{3}{2}\sin v\sin 2s, \frac{1}{2\sqrt{2}}\cos 2s+\frac{3}{2}\sin v\cos 2s,\\ & \frac{1}{2\sqrt{2}}\sin 3s+\frac{3}{2}\cos v\sin 3s),\end{aligned} \tag{3.30}$$

$$\begin{aligned}\varphi(s,v) = \quad & (4+\frac{1}{2\sqrt{2}}\sin 2s+\frac{3}{2}\sin v\sin 2s, \frac{1}{2\sqrt{2}}\cos 2s+\frac{3}{2}\sin v\cos 2s,\\ & 5-\frac{1}{2\sqrt{2}}\cos 3s-\frac{3}{2}\cos v\cos 3s),\end{aligned} \tag{3.31}$$

$$\begin{aligned}\varphi(s,v) = \quad & (4+\frac{1}{2\sqrt{2}}\sin 2s+\frac{3}{2}\sin v\sin 2s, \frac{1}{2\sqrt{2}}\sin 3s+\frac{3}{2}\cos v\sin 3s,\\ & 5-\frac{1}{2\sqrt{2}}\cos 3s-\frac{3}{2}\cos v\cos 3s),\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}\varphi(s,v) = \quad & (\frac{1}{2\sqrt{2}}\cos 2s+\frac{3}{2}\sin v\cos 2s, \frac{1}{2\sqrt{2}}\sin 3s+\frac{3}{2}\cos v\sin 3s,\\ & 5-\frac{1}{2\sqrt{2}}\cos 3s-\frac{3}{2}\cos v\cos 3s),\end{aligned} \tag{3.33}$$

respectively. Figure 2 illustrates the projected images of the hypersurface parametrized by $\varphi(s,0,v)$, as given in Eq. (3.29), together with the associated center curve. These visualizations make it possible to observe the influence of the parameter v on the spatial configuration of the hypersurface in the $xyz-, xyt-, xzt-$ and $yzt-$subspaces.

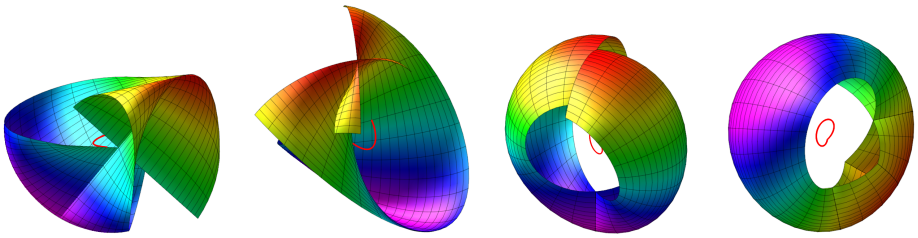


**Figure 2.** Projections of the directional tubular hypersurface $\varphi(s,0,v)$ onto the $xyz-$, $xyt-$, $xzt-$, and $yzt$-spaces.

Finally, by fixing $s=0$ , a parametrization $\varphi(0,u,v)$ of the hypersurface is obtained as

$$\varphi(0,u,v) = \quad (4+\frac{3}{2\sqrt{2}}\cos v\sin u, \frac{1}{2\sqrt{2}}+\frac{3}{2}\sin v, \frac{3}{2\sqrt{2}}\sin u\cos v, 5-\frac{1}{2\sqrt{2}}-\frac{3}{2}\cos u\cos v). \tag{3.34}$$

The corresponding projections of $\varphi(0,u,v)$ onto the $xyz-, xyt-, xzt-$ and $yzt-$subspaces are given in Eqs. (3.35)-(3.38)

$$\varphi(u,v) = \quad (4+\frac{3}{2\sqrt{2}}\cos v\sin u, \frac{1}{2\sqrt{2}}+\frac{3}{2}\sin v, \frac{3}{2\sqrt{2}}\sin u\cos v), \tag{3.35}$$

$$\varphi(u,v) = \quad (4+\frac{3}{2\sqrt{2}}\cos v\sin u, \frac{1}{2\sqrt{2}}+\frac{3}{2}\sin v, 5-\frac{1}{2\sqrt{2}}-\frac{3}{2}\cos u\cos v), \tag{3.36}$$

$$\varphi(u,v) = \quad (4+\frac{3}{2\sqrt{2}}\cos v\sin u, \frac{3}{2\sqrt{2}}\sin u\cos v, 5-\frac{1}{2\sqrt{2}}-\frac{3}{2}\cos u\cos v), \tag{3.37}$$

and

$$\varphi(u,v) = \quad (\frac{1}{2\sqrt{2}}+\frac{3}{2}\sin v, \frac{3}{2\sqrt{2}}\sin u\cos v, 5-\frac{1}{2\sqrt{2}}-\frac{3}{2}\cos u\cos v), \tag{3.38}$$

respectively. The projected images of $\varphi(0,u,v)$, as indicated in Eq. (3.34) are displayed in Figure 3. These plots highlight the cross-sectional behavior of the hypersurface and provide further visual evidence of the geometric effects induced by the quasi-frame construction onto the $xyz-, xyt-, xzt-$ and $yzt-$subspaces.

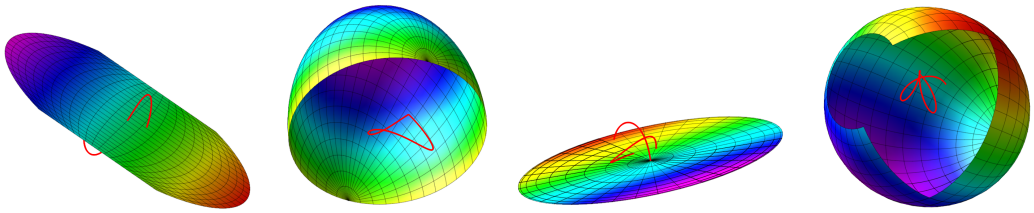


**Figure 3.** Projections of the directional tubular hypersurface $\varphi(0,u,v)$ onto the xyz-, xyt-, xzt-, and yzt-spaces

**Example 3.2.** A further example is provided below to deepen the analysis of directional tubular hypersurfaces in four-dimensional Euclidean space $\mathbb{E}^4$ within the quasi-frame. Consider a space curve

$$\begin{aligned}\alpha(s) = \; & (\sqrt{s}\sin\sqrt{s}+\cos\sqrt{s}, \frac{1}{2}-\sqrt{s}\cos\sqrt{s}+\sin\sqrt{s},\\ & \sqrt{s}\sin\sqrt{3s}+\frac{1}{\sqrt{3}}\cos\sqrt{3s}, \frac{3}{5}-\sqrt{s}\cos\sqrt{3s}+\frac{1}{\sqrt{3}}\sin\sqrt{3s})\end{aligned} \tag{3.39}$$

which is taken as the centre curve of the directional tubular hypersurface in $\mathbb{E}^4$. It can readily be verified that $\|\alpha'(s)\|=1$, which shows that the curve $\alpha(s)$ is parameterized by the arc length. Then, by means of the relations in Eq. (2.1), the quasi-frame vectors $\{\mathbf{T},\mathbf{N}_q,\mathbf{B}_q,\mathbf{C}_q\}$ associated with $\alpha(s)$ are obtained as follows:

$$\begin{aligned}\mathbf{T}(s) = \; & (\frac{1}{2}\cos\sqrt{s}, \frac{1}{2}\sin\sqrt{s}, \frac{\sqrt{3}}{2}\cos\sqrt{3s}, \frac{\sqrt{3}}{2}\sin\sqrt{3s})\\ \mathbf{N}_q(s) = \; & (0,0,\sin\sqrt{3s},-\cos\sqrt{3s})\\ \mathbf{B}_q(s) = \; & (\frac{-3(\sqrt{s}\cos\sqrt{s}+\frac{2}{3}\sin\sqrt{t})}{2\sqrt{1+3s}}, \frac{-3(\sqrt{s}\sin\sqrt{s}-\frac{2}{3}\cos\sqrt{s})}{2\sqrt{1+3s}}, \frac{\sqrt{3}\cos\sqrt{3s}}{2\sqrt{t+3s^2}}, \frac{\sqrt{3}\sin\sqrt{3s}}{2\sqrt{t+3s^2}})\\ \mathbf{C}_q(s) = \; & (\frac{-\sqrt{3}(\sqrt{s}\sin\sqrt{s}-\frac{1}{2}\cos\sqrt{s})}{\sqrt{1+3s}}, \frac{\sqrt{3}(2\sqrt{s}\cos\sqrt{s}+\sin\sqrt{s})}{2\sqrt{1+3s}}, \frac{-\cos\sqrt{3s}}{2\sqrt{1+3s}}, \frac{-\sin\sqrt{3s}}{2\sqrt{1+3s}})\end{aligned} \tag{3.40}$$

The quasi-curvatures associated with this frame are obtained from Eq. (2.3) and can be written as

$$k_1(s)=-\frac{3}{4\sqrt{s}},\quad k_2(s)=\frac{1}{4\sqrt{t+3s^2}},\quad k_3(s)=\frac{3}{4\sqrt{1+3s}}\quad\text{and}\quad k_4(s)=\frac{-3\sqrt{3}(s+1)}{\sqrt{4s+12s^2}}. \tag{3.41}$$

Taking equations (3.39) and (3.40) into equation (3.2), the directional tubular hypersurface is parametrized as

$$\begin{aligned}\varphi(s,u,v) = \; & (\sqrt{s}\sin\sqrt{s}+\cos\sqrt{s}-\tfrac{3r\sin u\cos v(\sqrt{s}\cos\sqrt{s}+\frac{2}{3}\sin\sqrt{s})}{2\sqrt{1+3s}}-\tfrac{\sqrt{3}r\sin v(\sqrt{s}\sin\sqrt{s}-\frac{1}{2}\cos\sqrt{s})}{\sqrt{1+3s}},\\ & \tfrac{1}{2}-\sqrt{s}\cos\sqrt{s}+\sin\sqrt{s}-\tfrac{3r\sin u\cos v(\sqrt{s}\sin\sqrt{s}-\frac{2}{3}\cos\sqrt{s})}{2\sqrt{1+3s}}-\tfrac{\sqrt{3}r\sin v(2\sqrt{s}\cos\sqrt{s}+\sin\sqrt{s})}{2\sqrt{1+3s}},\\ & \sin\sqrt{3s}(\sqrt{s}+r\cos u\cos v)+\cos\sqrt{3s}(\tfrac{1}{\sqrt{3}}+\tfrac{\sqrt{3}r\sin u\cos v}{2\sqrt{t+3s^2}}-\tfrac{r\sin v}{2\sqrt{1+3s}}),\\ & \tfrac{3}{5}-\cos\sqrt{3s}(\sqrt{s}+r\cos u\cos v)+\sin\sqrt{3s}(\tfrac{1}{\sqrt{3}}+\tfrac{\sqrt{3}r\sin u\cos v}{2\sqrt{t+3s^2}}-\tfrac{r\sin v}{2\sqrt{1+3s}})).\end{aligned} \tag{3.42}$$

For $r=2$, the parametrization of directional tubular hypersurface is

$$\begin{aligned}\varphi(s,u,v) = \; & (\sqrt{s}\sin\sqrt{s}+\cos\sqrt{s}-\tfrac{3\sin u\cos v(\sqrt{s}\cos\sqrt{s}+\frac{2}{3}\sin\sqrt{s})}{\sqrt{1+3s}}-\tfrac{2\sqrt{3}\sin v(\sqrt{s}\sin\sqrt{s}-\frac{1}{2}\cos\sqrt{s})}{\sqrt{1+3s}},\\ & \tfrac{1}{2}-\sqrt{s}\cos\sqrt{s}+\sin\sqrt{s}-\tfrac{3\sin u\cos v(\sqrt{s}\sin\sqrt{s}-\frac{2}{3}\cos\sqrt{s})}{\sqrt{1+3s}}-\tfrac{\sqrt{3}\sin v(2\sqrt{s}\cos\sqrt{s}+\sin\sqrt{s})}{\sqrt{1+3s}},\\ & \sin\sqrt{3s}(\sqrt{s}+2\cos u\cos v)+\cos\sqrt{3s}(\tfrac{1}{\sqrt{3}}+\tfrac{\sqrt{3}\sin u\cos v}{\sqrt{t+3s^2}}-\tfrac{\sin v}{\sqrt{1+3s}}),\\ & \tfrac{3}{5}-\cos\sqrt{3s}(\sqrt{s}+2\cos u\cos v)+\sin\sqrt{3s}(\tfrac{1}{\sqrt{3}}+\tfrac{\sqrt{3}\sin u\cos v}{\sqrt{t+3s^2}}-\tfrac{\sin v}{\sqrt{1+3s}})).\end{aligned} \tag{3.43}$$

By setting $v = 0$ in the general parametrization, given by Eq. (3.43), of the directional tubular hypersurface, one obtains the following parametrrization:

$$\begin{aligned}\varphi(s,u,0) = \quad & (\sqrt{s}\sin\sqrt{s} + \cos\sqrt{s} - \frac{3\sin u(\sqrt{s}\cos\sqrt{s} + \frac{2}{3}\sin\sqrt{s})}{\sqrt{1+3s}},\\ & \frac{1}{2} - \sqrt{s}\cos\sqrt{s} + \sin\sqrt{s} - \frac{3\sin u(\sqrt{s}\sin\sqrt{s} - \frac{2}{3}\cos\sqrt{s})}{\sqrt{1+3s}},\\ & \sqrt{s}\sin\sqrt{3s} + \frac{1}{\sqrt{3}}\cos\sqrt{3s} + 2\cos u\sin\sqrt{3s} + \frac{\sqrt{3}\sin u\cos\sqrt{3s}}{\sqrt{t+3s^2}},\\ & \frac{3}{5} - \sqrt{s}\cos\sqrt{3s} + \frac{\sin\sqrt{3s}}{\sqrt{3}} - 2\cos u\cos\sqrt{3s} + \frac{\sqrt{3}\sin u\sin\sqrt{3s}}{\sqrt{s+3s^2}}).\end{aligned} \tag{3.44}$$

From this parametrization, the hypersurface projections onto the three dimentional subspace can be derived directly. In particular, Eq. (3.44) gives the xyz-projection, whereas Eqs. (3.45)-(3.48) provide the projections onto the $xyz-, xyt-, xzt-$, and $yzt-$subspaces

$$\begin{aligned}\varphi(s,u) = \quad & (\sqrt{s}\sin\sqrt{s} + \cos\sqrt{s} - \frac{3\sin u(\sqrt{s}\cos\sqrt{s} + \frac{2}{3}\sin\sqrt{s})}{\sqrt{1+3s}},\\ & \frac{1}{2} - \sqrt{s}\cos\sqrt{s} + \sin\sqrt{s} - \frac{3\sin u(\sqrt{s}\sin\sqrt{s} - \frac{2}{3}\cos\sqrt{s})}{\sqrt{1+3s}},\\ & \sqrt{s}\sin\sqrt{3s} + \frac{1}{\sqrt{3}}\cos\sqrt{3s} + 2\cos u\sin\sqrt{3s} + \frac{\sqrt{3}\sin u\cos\sqrt{3s}}{\sqrt{s+3s^2}}),\end{aligned} \tag{3.45}$$

$$\begin{aligned}\varphi(s,u) = \quad & (\sqrt{s}\sin\sqrt{s} + \cos\sqrt{s} - \frac{3\sin u(\sqrt{s}\cos\sqrt{s} + \frac{2}{3}\sin\sqrt{s})}{\sqrt{1+3s}},\\ & \frac{1}{2} - \sqrt{s}\cos\sqrt{s} + \sin\sqrt{s} - \frac{3\sin u(\sqrt{s}\sin\sqrt{s} - \frac{2}{3}\cos\sqrt{s})}{\sqrt{1+3s}},\\ & \frac{3}{5} - \sqrt{s}\cos\sqrt{3s} + \frac{\sin\sqrt{3s}}{\sqrt{3}} - 2\cos u\cos\sqrt{3s} + \frac{\sqrt{3}\sin u\sin\sqrt{3s}}{\sqrt{s+3s^2}}),\end{aligned} \tag{3.46}$$

$$\begin{aligned}\varphi(s,u) = \quad & (\sqrt{s}\sin\sqrt{s} + \cos\sqrt{s} - \frac{3\sin u(\sqrt{s}\cos\sqrt{s} + \frac{2}{3}\sin\sqrt{s})}{\sqrt{1+3s}},\\ & \sqrt{s}\sin\sqrt{3s} + \frac{1}{\sqrt{3}}\cos\sqrt{3s} + 2\cos u\sin\sqrt{3s} + \frac{\sqrt{3}\sin u\cos\sqrt{3s}}{\sqrt{s+3s^2}},\\ & \frac{3}{5} - \sqrt{s}\cos\sqrt{3s} + \frac{\sin\sqrt{3s}}{\sqrt{3}} - 2\cos u\cos\sqrt{3s} + \frac{\sqrt{3}\sin u\sin\sqrt{3s}}{\sqrt{s+3s^2}}),\end{aligned} \tag{3.47}$$

and

$$\begin{aligned}\varphi(s,u) = \quad & (\frac{1}{2} - \sqrt{s}\cos\sqrt{s} + \sin\sqrt{s} - \frac{3\sin u(\sqrt{s}\sin\sqrt{s} - \frac{2}{3}\cos\sqrt{s})}{\sqrt{1+3s}},\\ & \sqrt{s}\sin\sqrt{3s} + \frac{1}{\sqrt{3}}\cos\sqrt{3s} + 2\cos u\sin\sqrt{3s} + \frac{\sqrt{3}\sin u\cos\sqrt{3s}}{\sqrt{s+3s^2}},\\ & \frac{3}{5} - \sqrt{s}\cos\sqrt{3s} + \frac{\sin\sqrt{3s}}{\sqrt{3}} - 2\cos u\cos\sqrt{3s} + \frac{\sqrt{3}\sin u\sin\sqrt{3s}}{\sqrt{s+3s^2}}).\end{aligned} \tag{3.48}$$

respectively. The projections of the hypersurface generated by $\varphi(s,u,0)$, as defined in Eq. (3.44), are depicted in Figure 4 in the $xyz-, xyt-, xzt-$ and $yzt-$subspaces, along with the curve $\alpha$.

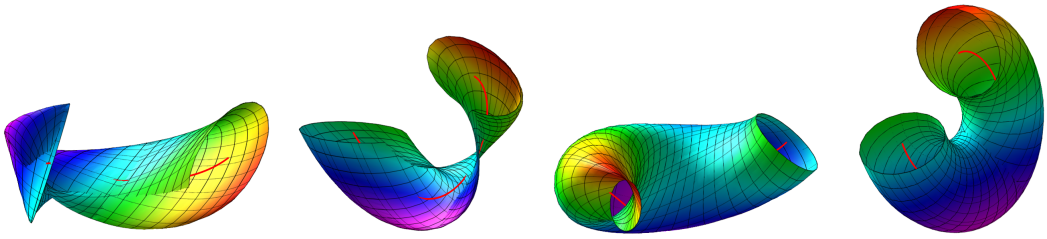


**Figure 4.** Projections of the directional tubular hypersurface $\varphi(s,u,0)$ onto the $xyz-$, $xyt-$, $xzt-$, and $yzt-$spaces.

To explore another class of sectional profiles of the hypersurface, we impose the restriction $u=0$ and consider the resulting parametrization $\varphi(s,0,v)$. In this case, the general parametrization takes the following reduced form:

$$\begin{aligned}\varphi(s,0,v) = \quad & (\sqrt{s}\sin\sqrt{s}+\cos\sqrt{s}-\frac{2\sqrt{3}\sin v(\sqrt{s}\sin\sqrt{s}-\frac{1}{2}\cos\sqrt{s})}{\sqrt{1+3s}},\\ & \frac{1}{2}-\sqrt{s}\cos\sqrt{s}+\sin\sqrt{s}-\frac{\sqrt{3}\sin v(2\sqrt{s}\cos\sqrt{s}+\sin\sqrt{s})}{\sqrt{1+3s}},\\ & \sqrt{s}\sin\sqrt{3s}+\frac{1}{\sqrt{3}}\cos\sqrt{3s}+2\cos v\sin\sqrt{3s}-\frac{\sin v\cos\sqrt{3s}}{\sqrt{1+3s}},\\ & \frac{3}{5}-\sqrt{s}\cos\sqrt{3s}+\frac{\sin\sqrt{3s}}{\sqrt{3}}-2\cos v\cos\sqrt{3s}-\frac{\sin v\sin\sqrt{3s}}{\sqrt{1+3s}}).\end{aligned} \tag{3.49}$$

Eqs. (3.50)-(3.53) describe the projections of $\varphi(s,0,v)$ onto the $xyz, xyt-, xzt-$ and $yzt-$subspaces

$$\begin{aligned}\varphi(s,v) = \quad & (\sqrt{s}\sin\sqrt{s}+\cos\sqrt{s}-\frac{2\sqrt{3}\sin v(\sqrt{s}\sin\sqrt{s}-\frac{1}{2}\cos\sqrt{s})}{\sqrt{1+3s}},\\ & \frac{1}{2}-\sqrt{s}\cos\sqrt{s}+\sin\sqrt{s}-\frac{\sqrt{3}\sin v(2\sqrt{s}\cos\sqrt{s}+\sin\sqrt{s})}{\sqrt{1+3s}},\\ & \sqrt{s}\sin\sqrt{3s}+\frac{1}{\sqrt{3}}\cos\sqrt{3s}+2\cos v\sin\sqrt{3s}-\frac{\sin v\cos\sqrt{3s}}{\sqrt{1+3s}}),\end{aligned} \tag{3.50}$$

$$\begin{aligned}\varphi(s,v) = \quad & (\sqrt{s}\sin\sqrt{s}+\cos\sqrt{s}-\frac{2\sqrt{3}\sin v(\sqrt{s}\sin\sqrt{s}-\frac{1}{2}\cos\sqrt{s})}{\sqrt{1+3s}},\\ & \frac{1}{2}-\sqrt{s}\cos\sqrt{s}+\sin\sqrt{s}-\frac{\sqrt{3}\sin v(2\sqrt{s}\cos\sqrt{s}+\sin\sqrt{s})}{\sqrt{1+3s}},\\ & \frac{3}{5}-\sqrt{s}\cos\sqrt{3s}+\frac{\sin\sqrt{3s}}{\sqrt{3}}-2\cos v\cos\sqrt{3s}-\frac{\sin v\sin\sqrt{3s}}{\sqrt{1+3s}}),\end{aligned} \tag{3.51}$$

$$\begin{aligned}\varphi(s,v) = \quad & (\sqrt{s}\sin\sqrt{s}+\cos\sqrt{s}-\frac{2\sqrt{3}\sin v(\sqrt{s}\sin\sqrt{s}-\frac{1}{2}\cos\sqrt{s})}{\sqrt{1+3s}},\\ & \sqrt{s}\sin\sqrt{3s}+\frac{1}{\sqrt{3}}\cos\sqrt{3s}+2\cos v\sin\sqrt{3s}-\frac{\sin v\cos\sqrt{3s}}{\sqrt{1+3s}},\\ & \frac{3}{5}-\sqrt{s}\cos\sqrt{3s}+\frac{\sin\sqrt{3s}}{\sqrt{3}}-2\cos v\cos\sqrt{3s}-\frac{\sin v\sin\sqrt{3s}}{\sqrt{1+3s}}),\end{aligned} \tag{3.52}$$

and

$$\begin{aligned}\varphi(s,v) = \quad & (\frac{1}{2} - \sqrt{s}\cos\sqrt{s} + \sin\sqrt{s} - \frac{\sqrt{3}\sin v(2\sqrt{s}\cos\sqrt{s} + \sin\sqrt{s})}{\sqrt{1+3s}}, \\ & \sqrt{s}\sin\sqrt{3s} + \frac{1}{\sqrt{3}}\cos\sqrt{3s} + 2\cos v\sin\sqrt{3s} - \frac{\sin v\cos\sqrt{3s}}{\sqrt{1+3s}}, \\ & \frac{3}{5} - \sqrt{s}\cos\sqrt{3s} + \frac{\sin\sqrt{3s}}{\sqrt{3}} - 2\cos v\cos\sqrt{3s} - \frac{\sin v\sin\sqrt{3s}}{\sqrt{1+3s}}),\end{aligned} \tag{3.53}$$

respectively. In Figure 5, the hypersurface generated by the parametrization $\varphi(s,0,v)$ in Eq. (3.49) is displayed through its projections onto the $xyz-$, $xyt-$, $xzt-$ and $yzt-$subspaces, together with the associated centre curve. These projected views reveal the role of the parameter $v$ in shaping the three dimentional Euclidean geometry of the directional tubular hypersurface.

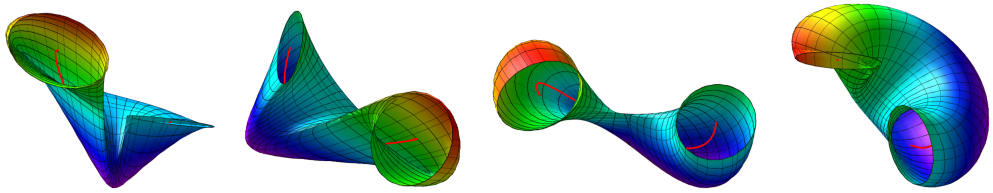


**Figure 5.** Projections of the directional tubular hypersurface $\varphi(s,0,v)$ onto the $xyz-$, $xyt-$, $xzt-$, and $yzt-$spaces.

Finally, by fixing $s=\pi^2$ , a parametrization $\varphi(\pi^2,u,v)$ of the hypersurface is obtained as

$$\begin{aligned}\varphi(\pi^2,u,v) = \quad & (-1 + \frac{3\pi\sin u\cos v}{\sqrt{1+3\pi^2}} - \frac{\pi\sqrt{3}\sin v}{\sqrt{1+3\pi^2}}, \frac{1}{2} + \pi - \frac{2\sin u\cos v}{\sqrt{1+3\pi^2}} + \frac{2\pi\sqrt{3}\sin v}{\sqrt{1+3\pi^2}}, \\ & \pi\sin\sqrt{3}\pi + \frac{1}{\sqrt{3}}\cos\sqrt{3}\pi + 2\cos u\cos v\sin\sqrt{3}\pi \\ & + \frac{\sqrt{3}\sin u\cos v\cos\sqrt{3}\pi}{\sqrt{\pi^2+3\pi^4}} - \frac{\sin v\cos\sqrt{3}\pi}{\sqrt{1+3\pi^2}}, \\ & \frac{3}{5} + \pi\cos\sqrt{3}\pi + \frac{1}{\sqrt{3}}\sin\sqrt{3}\pi - 2\cos u\cos v\cos\sqrt{3}\pi \\ & + \frac{\sqrt{3}\sin u\cos v\sin\sqrt{3}\pi}{\sqrt{\pi^2+3\pi^4}} - \frac{\sin v\sin\sqrt{3}\pi}{\sqrt{1+3\pi^2}}).\end{aligned} \tag{3.54}$$

The images of $\varphi(\pi^2,u,v)$ under projection onto the three-dimensional $xyz-$, $xyt-$, $xzt-$ and $yzt-$subspaces are described in Eqs. (3.55)-(3.58)

$$\begin{aligned}\varphi(u,v) = \quad & (-1 + \frac{3\pi\sin u\cos v}{\sqrt{1+3\pi^2}} - \frac{\pi\sqrt{3}\sin v}{\sqrt{1+3\pi^2}}, \frac{1}{2} + \pi - \frac{2\sin u\cos v}{\sqrt{1+3\pi^2}} + \frac{2\pi\sqrt{3}\sin v}{\sqrt{1+3\pi^2}}, \\ & \pi\sin\sqrt{3}\pi + \frac{1}{\sqrt{3}}\cos\sqrt{3}\pi + 2\cos u\cos v\sin\sqrt{3}\pi \\ & + \frac{\sqrt{3}\sin u\cos v\cos\sqrt{3}\pi}{\sqrt{\pi^2+3\pi^4}} - \frac{\sin v\cos\sqrt{3}\pi}{\sqrt{1+3\pi^2}}),\end{aligned} \tag{3.55}$$

$$\begin{aligned}\varphi(u,v) = \quad & (-1 + \frac{3\pi\sin u\cos v}{\sqrt{1+3\pi^2}} - \frac{\pi\sqrt{3}\sin v}{\sqrt{1+3\pi^2}}, \frac{1}{2} + \pi - \frac{2\sin u\cos v}{\sqrt{1+3\pi^2}} + \frac{2\pi\sqrt{3}\sin v}{\sqrt{1+3\pi^2}}, \\ & \frac{3}{5} + \pi\cos\sqrt{3}\pi + \frac{1}{\sqrt{3}}\sin\sqrt{3}\pi - 2\cos u\cos v\cos\sqrt{3}\pi \\ & + \frac{\sqrt{3}\sin u\cos v\sin\sqrt{3}\pi}{\sqrt{\pi^2+3\pi^4}} - \frac{\sin v\sin\sqrt{3}\pi}{\sqrt{1+3\pi^2}}),\end{aligned} \tag{3.56}$$

$$\begin{aligned}\varphi(u,v) = \; & (-1+\frac{3\pi\sin u\cos v}{\sqrt{1+3\pi^2}}-\frac{\pi\sqrt{3}\sin v}{\sqrt{1+3\pi^2}},\\ & \pi\sin\sqrt{3}\pi+\frac{1}{\sqrt{3}}\cos\sqrt{3}\pi+2\cos u\cos v\sin\sqrt{3}\pi\\ & +\frac{\sqrt{3}\sin u\cos v\cos\sqrt{3}\pi}{\sqrt{\pi^2+3\pi^4}}-\frac{\sin v\cos\sqrt{3}\pi}{\sqrt{1+3\pi^2}},\\ & \frac{3}{5}+\pi\cos\sqrt{3}\pi+\frac{1}{\sqrt{3}}\sin\sqrt{3}\pi-2\cos u\cos v\cos\sqrt{3}\pi\\ & +\frac{\sqrt{3}\sin u\cos v\sin\sqrt{3}\pi}{\sqrt{\pi^2+3\pi^4}}-\frac{\sin v\sin\sqrt{3}\pi}{\sqrt{1+3\pi^2}}),\end{aligned} \tag{3.57}$$

and

$$\begin{aligned}\varphi(u,v) = \; & (\frac{1}{2}+\pi-\frac{2\sin u\cos v}{\sqrt{1+3\pi^2}}+\frac{2\pi\sqrt{3}\sin v}{\sqrt{1+3\pi^2}},\\ & \pi\sin\sqrt{3}\pi+\frac{1}{\sqrt{3}}\cos\sqrt{3}\pi+2\cos u\cos v\sin\sqrt{3}\pi\\ & +\frac{\sqrt{3}\sin u\cos v\cos\sqrt{3}\pi}{\sqrt{\pi^2+3\pi^4}}-\frac{\sin v\cos\sqrt{3}\pi}{\sqrt{1+3\pi^2}},\\ & \frac{3}{5}+\pi\cos\sqrt{3}\pi+\frac{1}{\sqrt{3}}\sin\sqrt{3}\pi-2\cos u\cos v\cos\sqrt{3}\pi\\ & +\frac{\sqrt{3}\sin u\cos v\sin\sqrt{3}\pi}{\sqrt{\pi^2+3\pi^4}}-\frac{\sin v\sin\sqrt{3}\pi}{\sqrt{1+3\pi^2}}).\end{aligned} \tag{3.58}$$

respectively. The projected configurations associated with $\varphi(\pi^2,u,v)$, given in Eq. (3.54), are shown in Figure 6 in the $xyz-, xyt-, xzt-$ and $yzt-$subspaces.

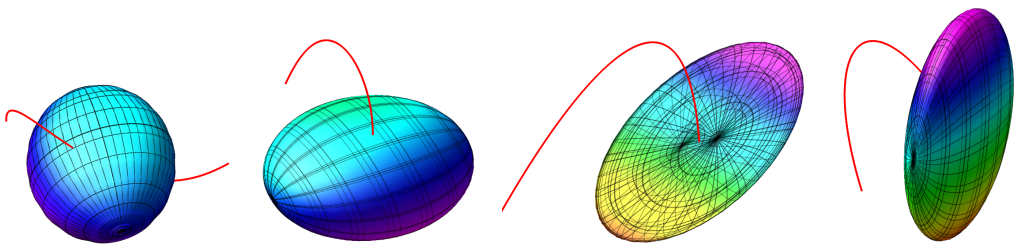


**Figure 6.** Projections of the directional tubular hypersurface $\varphi(\pi^2,u,v)$ onto the $xyz-, xyt-, xzt-$, and $yzt-$spaces

## 4. Conclusion

In this study, the tubular hypersurfaces in 4-dimensional Euclidean space with respect to quasi-frame have been investigated. A general parametrization for directional tubular hypersurfaces was introduced, and their differential geometric properties were examined in detail. Also, the unit normal vector field, fundamental form and shape operator matrices, Gaussian and mean curvature, and principal curvatures were obtained of this surface are clearly given. Moreover, the theoretical development was further reinforced by two illustrative examples, through which a more concrete understanding of the geometric structure of directional tubular hypersurfaces was achieved. In each case, the relevant curvatures were derived explicitly, and the associated directional tubular hypersurfaces were visualized via their projections onto several three-dimensional subspaces, namely $xyz$, $xyt$, $xzt$, and $yzt$ in Figures 1-6. These examples highlight the practical constructions and analytic explorations of tubular hypersurfaces generated by quasi-vectors. These computations provide insight into the local and global geometric behavior of the hypersurface and extend existing results in the theory of canal and tubular hypersurfaces to a higher-dimensional context.

## Acknowledgements

We would like to express our sincere thanks to the editor and the reviewers for their helpful comments and suggestions.

## Funding

There is no funding for this work.

## Availability of data and materials

Not applicable.

## Competing interests

The authors declare that they have no competing interests.

## Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

## References

[1] Alessio, O.: *Diferential Geometry of Intersection Curves $R^4$ of Three Implicit Surfaces*. Computer Aided Geometric Design, **26**, 455-471, (2009).
[2] Altın, M., Kazan, A.: *Tubular Hypersurfaces according to Extended Darboux Frame Field of First Kind in $\mathbb{E}^4$*. Turkish Journal of Science, **7** (2), 75-84, (2022).
[3] Aydın, M. E., Rafael, L.: *Rotational Surfaces in $\mathbb{R}^4$ with New Approaches and Examples*. International Electronic Journal of Geometry, **17** (1), 97-105, (2024).
[4] Blaga, P. A.: *On Tubular surfaces in computer graphics*. Studia univ. Babes-Bolyai, Informatica, **2**, 81-90, (2005).
[5] Bloomenthal, J.: *Calculation of reference frames along a space curve, Graphics gems*. Academic Press Professional, Inc., San Diego, CA, 1990.
[6] Bulca, B., Arslan, K., Bayram, B., Öztürk, G.: *Canal surfaces in 4-dimensional Euclidean space*. An International Journal of Optimization and Control: Theories & Applications, **7** (1), 83–89, (2017).
[7] Büyükkütük, S., Kişi. I., Öztürk, G., Kişi, E.: *Parallel Hypersurfaces in $\mathbb{E}^4$ and Their Applications to Rotational Hypersurfaces*. Mathematics, **13**(22), (2025), 3684.
[8] Coquillart, S.: *Computing offsets of B-spline curves*. Computer-Aided Design, **19**, 305-309, (1987).
[9] Çelik, E., Ekici, C., Tozak, H.: *Curvatures of Translational Hypersurfaces with Quasi-Frame in 4-Dimensional Space*. In Proceedings of the 7th International Palandöken Scientific Studies Congress. Erzurum, Türkiye, 293-310, (2023).
[10] Dede, M., Ekici, C., Tozak, H.: *Directional Tubular Surfaces*. International Journal of Algebra, **9** (12), 527-535, (2015).
[11] Dede, M. Ekici, C., Görgülü, A.: *Directional q-frame along a space curve*. IJARCSSE, **5**, 775-780, (2015).
[12] Do Carmo, M. P.: Differential Geometry of Curves and Surfaces. Prentice-Hall, Englewood Cliffs, NJ, (1976).
[13] Ekici Coşkun, A., Akça, Z.: *The ruled surfaces generated by quasi-vectors in $\mathbb{E}^4$ space*. Hagia Sophia Journal of Geometry, **5** (2), 6-17, (2023).
[14] Elsayied, H. K., Tawfiq, A. M. Elsharkawy, A.: *Special Smarandach curves according to the quasi frame in 4-dimensional Euclidean space $\mathbb{E}^4$*. Houston J. Math, **74** (2), 467-482, (2021).
[15] Fastenakels, J.: *Ideal Tubular Hypersurfaces in Real Space Forms*. Archivum Mathematicum, **42** (3) (2006), 295-305.
[16] Ganchev, G. and Milousheva, V.: *On the theory of surfaces in the four-dimensional Euclidean space*. Kodai Math. J., **31** (2), 183-198, (2008).
[17] Gaspard, M.: Application de l'analyse a la géométrie. Bachelier, (1850).
[18] Gezer, B., Ekici, C.: *On space curve with quasi frame in $\mathbb{E}^4$*. 4th International Black Sea Modern Scientific Research Congress, 1951-1962, (2023).
[19] Gluck, H.: *Higher Curvatures of Curves in Euclidean Space*. The American Mathematical Monthly, **73** (7), 699-704, (1966).
[20] Gray, A.: Modern differential geometry of curves and surface. CRS Press, Inc, (1993).
[21] Kaymanlı, G. U., Ekici, C., Dede, M.: *Directional canal surfaces in $\mathbb{E}^3$*. ISMS2018, Ankara, 90-107, (2018).
[22] Kazan, A., Altın, M., Yoon, D. W.: *Geometric Characterizations of Canal Hypersurfaces in Euclidean Spaces*. Filomat, **37** (18), 5909-5920, (2021).
[23] Kazan, A., Altın, M.: *Some Curvatures of Tubular Hypersurfaces in $\mathbb{E}^4$*. ASES III. International Health, Engineering and Sciences Conference, Çanakkale, Türkiye, Conference Book, 56-63, (2022).
[24] Kazan, A., Altın, M., Turgay, N. C.: *Some classifications for Gauss map of Tubular hypersurfaces in $\mathbb{E}_1^4$ concerning linearized operators $L_k$*. Hacettepe Journal of Mathematics and Statistics, **54** (4), 1329-1344, (2025).
[25] Kızıltuğ, S., Dede, M., Ekici, C.: *Tubular surfaces with Darboux frame in Galilean 3-space*. Facta Universitatis, Series: Mathematics and Informatics, **34** (3), 461-471, (2019).
[26] Kim, Y. H., Liu, H., Qian, J.: *Some characterizations of canal surfaces*. Bulletin of the Korean Mathematical Society, **53** (2), 461-477, (2016).
[27] Kuhnel, W.: Differential geometry, curves-surfaces-manifolds. American Mathematical Society, (2002).
[28] Maekawa, T., Patrikalakis, N. M., Sakkalis, T., Yu, G.: *Analysis and applications of pipe surfaces*. Computer Aided Geometric Design, **15**, 437-458, (1998).
[29] O'Neill, B.: Elementary Differential Geometry. Academic Press, (1966).
[30] Otsuki, T., Shiohama, K.: *A theory of ruled surfaces in $\mathbb{E}^4$*. Kodai Math Sem. Rep., **19**, 370-380, (1967).

[31] Óláh-Gál, R., Pál, L.: *Some notes on drawing twofolds in 4-dimensional Euclidean space*. Acta Universitatis Sapientiae, Informatica, **1** (2), 125-134, (2009).
[32] Saji, K.: *Singularities of non-degenerate 2-ruled hypersurfaces in 4-space*. Hiroshima Math. J., **32**, 309-323, (2002).
[33] Shaw, R.: *Vector cross products in n dimensions*. International Journal of Mathematical Education In Science and Technology, **18** (6), 803-816, (1987).
[34] Shin, S. K. Yoo, S. K. Cho, S. K. Chung, W. H.: *Directional Offset of a Spatial Curve for Practical Engineering Design*. ICCSA, 711-720, (2003).
[35] Struik, D. J.: Lectures on Classical Differential Geometry. Addison-Wesley Publishing Company, Inc, (1961).
[36] Turgay, N. C., Sağdiç, M., Kılıç, E.: *Simple Rotational Surfaces in Euclidean 4-Space with Generalized 1-Type Gauss Map*. International Electronic Journal of Geometry, **18** (1), 48-59, (2025).
[37] Williams, M. Z., Stein, F. M.: *A Triple Product of Vectors in Four Space*. Mathematics Magazine, **37** (4), 230-235, (1964).
[38] Xu, Z., Feng, R., Sun, J.G.: *Analytic and algebraic properties of canal surfaces*. Journal of Computational and Applied Mathematics, **195**, 220-228, (2006).
[39] Yağbasan, B., Tozak, H., Ekici, C.: *Directional Tube Surface in Euclidean 4-Space*. Hagia Sophia Journal of Geometry, **5** (2), 18-30, (2023).
[40] Yağbasan, B., Ekici, C.: *Shape Operators of a Directional Tubular Surface in 4-space*. Eskişehir Teknik Üniversitesi Bilim ve Teknoloji Dergisi B-Teorik Bilimler, **13** (2), 109-121, (2025).
[41] Yıldırım, A.: *Tubular Surfaces According to a Focal Curve in* $\mathbb{E}^3$. Turkish Journal of Mathematics and Computer Science, **15** (2), 433-442, (2023).
[42] Yoon, D.W., Yuzbasi, Z.K.: *A method for hypersurface family with common geodesic curve in 4D Galilean space G4*. Journal of the Korean Society of Mathematical Education Series B: Pure and Applied Mathematics, **25**(4), 229-241, (2018).

## Affiliations

BAŞAK YAĞBASAN
**ADDRESS:** Eskişehir Osmangazi University, Graduate School of Natural and Applied Sciences, Eskişehir-Türkiye.
**E-MAIL:** bskyagbasan@gmail.com
**ORCID ID:0000-0003-4067-3034**

CUMALI EKICI
**ADDRESS:** Eskişehir Osmangazi University, Department of Mathematics and Computer Science, Eskişehir-Türkiye.
**E-MAIL:** cekici@ogu.edu.tr
**ORCID ID:0000-0002-3247-5727**

HATICE TOZAK
**ADDRESS:** MEB Denizli Provincial Directorate of National Education, Department of Strategy, Denizli-Türkiye.
**E-MAIL:** hatice.tozak@gmail.com
**ORCID ID:0000-0003-1383-0276**