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A note on Berwald eikonal equation

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Abstract. In this study, firstly, we generalize Berwald map by introducing the concept of a Riemannian map. After that we find Berwald eikonal equation through using the Berwald map. The eikonal equation of geometrical optic that examining light reflects, refracts at smooth, plane interfaces is obtained for Berwald condition.

1. Introduction

Finsler manifolds can be thought as generalization of Riemannian manifolds; tangent spaces carry Minkowski norms instead of inner products and geometric objects on tangent vectors depend not only on the base but also on the fibre component. Finsler manifold have an intrinsic geometrical significance and also they have been used to model a variety of problems from dynamics, optics and relativity, cf. eg. [4] and [8].

One of the relevant figures working on that period on Finsler geometry was L. Berwald, who introduced a connection and a class of spaces sharing his name [7]. Especially, positive definite Berwald manifolds constitute the conceptually simplest and the best understood class of Finsler manifolds.

The eikonal equation is frequently solved using characteristic. This involves solving the ray equations in a combined coordinate and ray parameter phase space and then integrating the traveltime along these rays [10]. Physically rays are the trajectories along which high-frequency energy flows. A number of efficient methods for solving the ray equations have been developed [3]. Using the method of characteristics, traveltimes were computed along rays and approximate solutions of the eikonal equation could also obtained by R. L. Nowack [9].

∇f is used many areas of science such as mathematical physics and geometry. For example, the Riemannian condition $\|\nabla f\|^2 = 1$ is precisely the eikonal equation of geometrical optics. In the geometrical optical interpretation, the level sets of f are interpreted as wave fronts. The characteristics of the eikonal equation are then the solutions of the gradient flow equation for f,

$$x' = \operatorname{grad} f(x) \tag{1}$$

which are geodesics of M orthogonal to the level sets of f, and which are parametrized by arc length. These geodesics can be interpreted as light rays orthogonal to the wave fronts [1].

The concept of a Riemannian map was introduced by Fischer and it is shown that these maps are solutions of the eikonal equation [1]. The notion of semi-Riemannian map was stated, using the map, the solution of the eikonal equation was obtained by Garcio et al [5].


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In this paper, we generalize these maps to Berwald manifolds by introducing the concept of a Riemannian map. We find Berwald eikonal equation though using the Berwald map.

2. Preliminaries

Let M be a connected, $n-$dimensional, C^∞ manifold and $TM = \cup_{x\in M} T_xM$ be the tangent bundle of M, where T_xM is the tangent space at $x \in M$. We denote a typical point in TM by (x, y). Set $TM_0 = TM \setminus \{0\}$, where $\{0\}$ stands for $\{(x, 0) : x \in X,\ 0 \in T_xM\}$. A Finsler metric on M is a function $F : V \to [0,\ \infty)$ with the following properties:

i) F is C^∞ on TM_0.

ii) At each point $x \in M$, the restriction $F_x : F\mid_{T_xM}$ is a Minkowski norm on T_xM. So the pair $(M,\ F)$ is called a Finsler manifold [11].

A Finsler metric F on a manifold M is called Berwald metric if in a standard local coordinate system $(x^i,\ y^i)$ in TM_0, the Christoffel symbols $\Gamma^i_{jk} = \Gamma^i_{jk}(x)$ are functions of $x \in M$ only, in which case, $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ are quadratics in $y = y^i\frac{\partial}{\partial x^i}\mid_x$. Riemannian metrics and Minkowski metrics are trivial Berwald metrics.

The Legendre transformation $l : TM \to T^*M$ is defined as

$$l(Y) = \begin{cases} g_Y(Y, \cdot), & Y \neq 0 \\ 0, & Y = 0, \end{cases}$$

where g is the symmetric bilinear form.

Let $f = M \to \mathbb{R}$ be a smooth function on M. The gradient of f is defined as $\nabla f = l^{-1}(df)$, where $l^{-1} : T^*M \to TM$ is the inverse Legendre transformation. Thus we have

$$df(X) = g_{\nabla f}(\nabla f,\ X),\ X \in TM. \tag{2}$$

[2].

Let M and M' be a Finsler manifolds with metric functions $F(x,\ y)$ and $F'(x',\ y')$. A differentiable map $f : M \to M'$ is local isometry if each of the tangent maps $f_* : T_xM \to T_{f(x)}M'$ is a vector space isomorphism which preserves the metric. A local isometry is an isometry if it is one to one and onto [6].

Let $(M,\ F)$ be a Berwald space. Given any parallel vector field V along a curve σ in M. We can express the Finslerian norm of V as

$$F(V) = \sqrt{g_V(V,\ V)} \tag{3}$$

where $g_V = g_{ij(\sigma,V)}dx^idx^j$ [4].

Let $f : (M,\ g_M) \to (N,\ g_N)$ be a smooth map between smooth finite dimensional Riemannian manifolds $(M,\ g_M)$ and $(N,\ g_N)$. Let

$$f_* : V_x \oplus H_x \to rangef_* \oplus range(f_*)^\perp$$

denote the tangent map at $x \in M$. Here $V_x = \ker f_* \subseteq T_xM$ denotes the vertical subspace of T_xM and $H_x = \ker(f_*)^\perp \subseteq T_xM$ denotes the horizontal subspace of T_xM. If the horizontal restriction $(f_*)^h : H_x \to rangef_*$ is a linear isometry between the inner product spaces $(H_x,\ g_M(x)\mid_{H_x})$ and $(rangef_*,\ g_N(y)\mid_{rangef_*})$, $y = f(x)$. The map is a Riemannian map if f is Riemannian at each $x \in M$ [1].

Theorem 2.1. Let $f : (M,\ g_M) \to (N,\ g_N)$ be a smooth map between smooth finite dimensional Riemannian manifolds $(M,\ g_M)$ and $(N,\ g_N)$, such that M is a connected manifold.

Then $rankf$ is constant on M and the norm squared $\|f_*\|^2$ of f_* satisfies

$$\|f_*\|^2 = rankf.$$

If in the above theorem $(N, g_N) = (\mathbb{R},\ 1) = \mathbb{R}$ is the real line with its Euclidean metric, and if $f : (M,\ g_M) \to \mathbb{R}$ is a real valued Riemannian map, then $rankf$ is constant on M equal to either zero or one. If $rankf = 0,$ then f is a constant map, and if $rankf = 1,$ then f satisfies the eikonal equation

$$\|f_*\|^2 = 1$$

of geometrical optics [1].

3. Berwald map and Berwald eikonal equation

Let M_1 and M_2 be differentiable, connected manifolds of dimensionals n_1 and n_2, respectively. Let $f : (M_1,\ F_1) \to (M_2,\ F_2)$ be a map, where F_1 and F_2 are Berwald metrics. Let $f_* : T_xTM_1 \to T_yTM_2$ denotes the tangent map at $x \in M_1,\ y = f(x) \in M_2.$ Then T_xTM_1 and T_yTM_2 split orthogonally as

$$T_xTM_1 = V_xTM_1 \oplus H_xTM_1$$

and

$$T_yTM_2 = rangef_* \oplus range(f_*)^{\perp}.$$

Here

$$V_xTM_1 = \ker f_* \subseteq T_xTM_1$$

denotes the vertical subspace of T_xTM_1 and

$$H_xTM_1 = \ker(f_*)^{\perp} \subseteq T_xTM_1$$

denotes the horizontal subspace of T_xTM_1. Thus, we have

$$f_* : V_xTM_1 \oplus H_xTM_1 \to rangef_* \oplus range(f_*)^{\perp}.$$

Using these expression, we can write the following definitions:

Definition 3.1. A smooth map $f : (M_1,\ F_1) \to (M_2,\ F_2)$ between smooth finite dimensional $(M_1,\ F_1)$ and $(M_2,\ F_2)$ is Berwald at $x \in M_1$ if the horizontal restriction

$$(f_*)^h : H_xTM_1 \to rangef_*$$

is a linear isometry between $(H_xTM_1,\ F_1(x)\mid_{H_x})$ and $(rangef_*,\ F_2(y)\mid_{rangef_*}),$ $y = f(x).$ The map is a Berwald map if f is Berwald at each $x \in M_1$.

Definition 3.2. Let $f : (M_1,\ F_1) \to (M_2,\ F_2)$ be a map. Then the square norm of f_* is the map $\|f_*\|^2 : M_1 \to \mathbb{R}^+$ defined by $\|f_*\|^2(p_1) = \|f_{*p_1}\|^2.$

Note that, since $\|f_*\|^2 = \sum_{i=1}^{n_1} F_1^2(X_i)F_2^2(f_*(X_i)),$ where $\{X_1, ..., X_{n_1}\}$ is an orthonormal local frame of TTM_1.

Let us compute the square norm of a map $f : (M,\ F) \to (\mathbb{R}^+,\ dt \otimes dt)$. For this, let $\{X_1, ..., X_n\}$ be an orthonormal local frame. We have

$$\|f_*\|^2 = \sum_{i=1}^{n} F^2(X_i)(dt \otimes dt)(f_*X_i,\ f_*X_i)$$

From here, using the expression (2), then we find

$$\begin{aligned} \|f_*\|^2 &= \sum_{i=1}^{n} F^2(X_i)(g_{\nabla f}(\nabla f,\ X_i))^2(dt \otimes dt)\left(\frac{d}{dt} \circ f,\ \frac{d}{dt} \circ f\right) \\ &= g_{\nabla f}(\nabla f,\ \nabla f) \end{aligned}$$

From the equation (3), we have $\|f_*\|^2 = F^2(\nabla f)$. The generalized eikonal equations for $f : (M,\ F) \to (\mathbb{R}^+,\ dt \otimes dt)$ become $\|f_*\| = F(\nabla f) = 1$. This extension enables a fast calculation of geodesic paths.

Remark 3.1. According to the Theorem 2.1, for Berwald, this case is a geometrical optical condition which it describes light propagation in terms of rays.

Lemma 3.1. Let $f : (M,\ F) \to (\mathbb{R}^+,\ dt \otimes dt)$ be a map. Then

$$\nabla(F^2(\nabla f)) = 2\nabla_{\nabla f}\nabla f,$$

where ∇f and $\nabla(F^2(\nabla f))$ denote the gradients of f and $F^2(\nabla f)$ on Berwald manifolds.

Proof. If $X \in \Gamma TM$, then we have

$$g(\nabla(F^2(\nabla f)),\ X) = 2g_{\nabla f}(\nabla_X \nabla f,\ \nabla f)$$

Since $h_f(X) = \nabla_X \nabla f$, we find

$$g(\nabla(F^2(\nabla f)),\ X) = 2g_{\nabla f}(\nabla_{\nabla f}\nabla f,\ X).$$

Hence, this completes the proof.

Proposition 3.1. Let $f : (M,\ F) \to (\mathbb{R}^+,\ dt \otimes dt)$ be a map. If f satisfies a Berwald eikonal equation $F^2(\nabla f) = 1$, then ∇f is a geodesic vector field on $(M,\ F)$.

Proof. For Berwald eikonal equation, we have $F^2(\nabla f) = 1$. From Lemma 3.1., we find

$$\nabla(F^2(\nabla f)) = 2\nabla_{\nabla f}\nabla f = 0. \tag{4}$$

Therefore, ∇f is a geodesic vector field on $(M,\ F)$. Namely, integral curves of the gradient flow of f are geodesics of the Berwald manifolds.

Proposition 3.2. Let $f : (M,\ F) \to (\mathbb{R}^+,\ dt \otimes dt)$ be a map. Then

i) The rank of f is constant on M.

ii) $F^2(\nabla f)$ is constant on M.

Proof. The rank of f is constant on M if and only if for every $p \in M$ either $\nabla f(p) = 0$ or $\nabla f(p) \neq 0$. Here, we will only notice that $\nabla f(p) \neq 0$ for every $p \in M$.

i) Let $p,\ q \in M$ and $\gamma : [a,\ b] \to M$ be a curve with $\gamma(a) = p,\ \gamma(b) = q$. Since ∇f is a parallel vector field on $(M,\ F)$, $(\nabla f) \circ \gamma$ is parallel vector field along γ. By the uniqueness of parallel vector fields along a curve with respect to initial condition, $\nabla f(p) \neq 0 \Leftrightarrow \nabla f(q) \neq 0$. Thus, we have $\nabla f(p) \neq 0$ for every $p \in M$.

ii) According to the expression (4), $F^2(\nabla f)$ is constant on M.

4. Conclusion

In this work, we introduce Berwald map and study to explain the eikonal equation of geometrical optic for Berwald condition. In future work, we will consider affine solution of Berwald eikonal equation. First of all, the physical implications of affine solution of Berwald eikonal equation should be investigated. Last, an obvious next step would be to show how to use this theorem where generalizations of Berwald metric are employed in order to construct the eikonal equation in pseudo-Finsler space.

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