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Simplicial and Crossed Resolutions of Commutative Algebras

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In this paper we investigate the relationship between simplicial and crossed resolutions of commutative algebras. © 1996 Academic Press, Inc.

INTRODUCTION

In his work on the homology of commutative algebras [1], André introduced a "step-by-step" construction of a resolution of an algebra. In many ways this is an algebra version of the construction of a CW-complex, or more exactly of an Eilenberg–Mac Lane space corresponding to a group. The resolution is built up so that at each stage the next step is formed by adding in new simplices to kill the homotopy groups of the previous step.

Simplicial resolutions were used by Illusie [10] to construct the cotangent complex of an algebra. In earlier treatments of this Lichtenbaum and Schlessinger [11] used an idea of Gerstenhaber to construct a special type of complex. Comparing this with results on crossed resolutions, in group cohomology theory, the second author showed how this corresponds to a crossed resolution of algebras; cf. [15].

In this paper we relate André's construction to an obvious construction of a crossed resolution of an algebra. This uses a neat description of the passage from simplicial algebras to crossed complexes analogous to that used by Carrasco [6] (cf. also Carrasco and Cegarra [7] and Ehlers and Porter [9]) in the group or groupoid case. The short proof that the construction does give a crossed complex is new. The main results are that

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this construction does give a ''step-by-step'' construction of a crossed resolution given one of a simplicial resolution and to give explicit description of this in low dimensions. The tools needed are the structure maps of Carrasco [6] and the higher order Peiffer elements discussed in our earlier paper [3] (cf. also [2]).

One of the aims of this series of papers is to show that what might be called ''combinatorial algebra theory,'' by analogy with ''combinatorial group theory,'' is an area with interesting structure and which contains potentially important new ideas.

1. PRELIMINARIES

Let **k** be a fixed commutative ring with $1 \neq 0$. All of the **k**-algebras discussed herein are assumed to be commutative and associative but we will want to consider ideals and modules to be algebras and so will not be requiring *algebras* to have unit elements. The category of commutative algebras will be denoted by **Alg**.

1.1. *Simplicial Algebras*

We refer the reader to the book ''Homologie des algèbres commutatives,'' by André [1], for more details on simplicial algebras.

A simplicial algebra **E** is a simplicial object in the category of algebras. We will denote the category of simplicial algebras by **SimpAlg**.

For any simplicial module **E**, there is an associated chain complex of **k**-modules. The module of n-chains is E_n, itself. The differentials ∂_n: $E_n \to E_{n-1}$ are defined by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i^n.$$

We will speak of the nth *homology module* $H_n(\mathbf{E})$ of the simplicial **k**-module **E**. This is, of course, defined by

$$H_n(\mathbf{E}) = \frac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_{n+1}}.$$

A simplicial algebra **E** is *augmented* by specifying an algebra E or equivalently a constant simplicial algebra $\mathbf{K}(E, 0)$, and a surjective **k**-algebra homomorphism, $f = d_0^0$: $E_0 \to E$ with $fd_0^1 = fd_1^1$: $E_1 \to E$. An *augmentation* of the simplicial algebra **E** is the resulting map $\mathbf{E} \to \mathbf{K}(E, 0)$. An augmented simplicial algebra is *acyclic* if the corresponding complex is

acyclic, i.e., $H_n(\mathbf{E}) \cong 0$ for $n > 0$ and $H_0(\mathbf{E}) \cong E$ with the latter isomorphism induced by f.

Let B be a commutative $\mathbf{k}$-algebra. A *free simplicial resolution* of B consists of a simplicial algebra $\mathbf{E}$ together with an augmentation f: $E_0 \to B$ such that $(\mathbf{E}, f)$ is acyclic and each E_n is free.

We will summarise André's construction of a simplicial resolution shortly.

The Moore complex and the homotopy module of a simplicial algebra. Recall that given a simplicial algebra $\mathbf{E}$, *the Moore complex* $(\mathbf{NE}, \partial)$ of $\mathbf{E}$ is the chain complex defined by

$$(\mathbf{NE})_n = \bigcap_{i=0}^{n-1} \operatorname{Ker} d_i^n$$

with ∂_n: $NE_n \to NE_{n-1}$ induced from d_n^n by restriction.

The nth *homotopy module* $\pi_n(\mathbf{E})$ of $\mathbf{E}$ is the nth homology of the Moore complex of $\mathbf{E}$, i.e.,

$$\begin{aligned}\pi_n(\mathbf{E}) &\cong H_n(\mathbf{NE}, \partial) \\ &= \bigcap_{i=0}^{n} \operatorname{Ker} d_i^n / d_{n+1}^{n+1}\left(\bigcap_{i=0}^{n} \operatorname{Ker} d_i^{n+1} \right).\end{aligned}$$

1.2. *Step-By-Step Constructions*

This section is a brief résumé of how to construct simplicial resolutions. The work depends heavily on a variety of sources, mainly [1, 13, 17]. The reader is referred to the book of André [1] for full details and more references.

First, some notation and terminology.

Let $[n]$ be the ordered set, $[n] = \{0 < 1 < \cdots < n\}$. We define the following maps: First, the injective monotone map δ_i^n: $[n-1] \to [n]$ is given by

$$\delta_i^n(x) = \begin{cases} x & \text{if} \quad x < i \\ x+1 & \text{if} \quad x \geq i \end{cases}$$

for $0 \leq i \leq n \neq 0$. On the other hand, an increasing surjective monotone map σ_i^n: $[n+1] \to [n]$ is given by

$$\sigma_i^n(x) = \begin{cases} x & \text{if} \quad x \leq i \\ x-1 & \text{if} \quad x > i \end{cases}$$

for $0 \leq i \leq n$. We denote by $\{m, n\}$ the set of increasing surjective maps $[m] \to [n]$; cf. [13].

1.2.1. Killing Elements in Homotopy Modules

This section describes the "step-by-step" construction of André [1].

Let $\mathbf{E}$ be a simplicial algebra and let $k \geq 1$ be fixed. Suppose we are given a set Ω of elements

$$\Omega = \{x_\lambda : \lambda \in \Lambda\},$$

$x_\lambda \in \pi_{k-1}(\mathbf{E})$; then we can choose a corresponding set of elements $w_\lambda \in NE_{k-1}$ so that

$$x_\lambda = w_\lambda + \partial_k(NE_k).$$

(If $k = 1$, then as $NE_0 = E_0$, the condition that $w_\lambda \in NE_0$ is empty.) We want to define a simplicial algebra, $\mathbf{F} = \mathbf{E}[\Omega]$ with a monomorphism $\mathbf{i}$: $\mathbf{E} \to \mathbf{F}$ such that

$$\pi_{k-1}(\mathbf{i}) : \pi_{k-1}(\mathbf{E}) \to \pi_{k-1}(\mathbf{F})$$

"kills off" the x_λ's. We do this by adding new indeterminates into NE_k to enlarge it so as to make $\mathbf{i}(w_\lambda) \in \partial NF_k$. More precisely,

(1) F_n is a free E_n-algebra,

$$F_n = E_n[y_{\lambda,t}] \qquad \text{with } \lambda \in \Lambda \text{ and } t \in \{n, k\}.$$

(2) For $0 \leq i \leq n$, the algebra homomorphism $s_i^n : F_n \to F_{n+1}$ is obtained from the homomorphism $s_i^n : E_n \to E_{n+1}$ with the relations

$$s_i^n(y_{\lambda,t}) = y_{\lambda,u} \qquad \text{with } u = t\sigma_i^n, t : [n] \to [k].$$

(3) For $0 \leq i \leq n \neq 0$, the algebra homomorphism $d_i^n : F_n \to F_{n-1}$ is obtained from $d_i^n : E_n \to E_{n-1}$ with the relations

$$d_i^n(y_{\lambda,t}) = \begin{cases} y_{\lambda,u} & \text{if the map } u = t\delta_i^n \text{ is surjective} \\ t'(w_\lambda) & \text{if } t\delta_i^n = \delta_k^k t' \\ 0 & \text{if } t\delta_i^n = \delta_j^k t' \text{ with } j \neq k \end{cases}$$

by extending linearly.

Here $t' : [n-1] \to [k-1]$. It thus corresponds to a unique algebra homomorphism $t' : E_{k-1} \to E_{n-1}$ (see André [1]).

We now examine this construction for a single element to see what it does:

EXAMPLE. Suppose $x \in \pi_1(\mathbf{E})$ (so $k = 2$). Pick a $w \in NE_1$ so that

$$x = \overline{w} = w + \partial_2(NE_2) \in \pi_1(\mathbf{E}).$$

We need a $y \in NE_2$ with

$$w = \partial(y) = d_2(y) \qquad \text{with } \overline{w} = w + \partial_2(NE_2) \in \pi_1(\mathbf{E})$$

and hence we add a new indeterminate y (which will be non-degenerate) into E_2 to form

$$F_2 = E_2[y] \qquad \text{with } d_0(y) = d_1(y) = 0 \text{ and } d_2(y) = w,$$

which implies

$$\mathbf{i}(\overline{w}) = \mathbf{i}(w + \partial NE_2) = 0,$$

as required. We cannot stop here as the images of y under s_0, s_1, s_2 are not yet defined.

For the next step we build F_3 so as to receive the degenerate images of y, i.e.,

$$F_3 = E_3[y_t],$$

the polynomial ring on a set of indeterminants $\{y_t\}$, where t: [3] $\to$ [2]. So there are three degenerate images corresponding to $s_0(y), s_1(y), s_2(y)$. We set

$$s_0(y) = y_{\sigma_0}, \qquad s_1(y) = y_{\sigma_1}, \qquad s_2(y) = y_{\sigma_2},$$

and also need to construct the face operators

$$d_0, d_1, d_2, d_3 \colon F_3 \to F_2$$

but these are determined in advance since

$$d_0 s_i(y) = s_{i-1} d_0(y) = 0 \qquad \text{unless } i = 0,$$

in which case $d_0 s_0(y) = y$. We then define recursively the higher dimensional images of y. In the formula given above this is done all together (following André [1]).

Remark. In the above "step-by-step" construction, we have the following properties:

(i) $F_n = E_n$ for $n < k$,

(ii) $F_k =$ a free E_k-algebra over a set of non-degenerate indeterminates, all of whose faces are zero except the kth,

(iii) F_n is a free E_n-algebra over the degenerate elements for $n > k$.

We have immediately the following result, as expected.

PROPOSITION 1.1. *The inclusion of simplicial algebras* $\mathbf{E} \hookrightarrow \mathbf{F}$, *where* $\mathbf{F} = \mathbf{E}[\Omega]$, *induces the homomorphism*

$$\pi_n(\mathbf{E}) \to \pi_n(\mathbf{F}).$$

For $n < k - 1$,

$$\pi_n(\mathbf{E}) \cong \pi_n(\mathbf{F})$$

and for $n = k - 1$, *this homomorphism is an epimorphism with kernel generated by elements of the form* $x_\lambda = w_\lambda + \partial_k N E_k$, *where* $\Omega = \{x_\lambda : \lambda \in \Lambda\}$.

1.2.2. Constructing Simplicial Resolutions

The following result is due to André [1].

THEOREM 1.2. *If* B *is a commutative* **k***-algebra, then it has a simplicial resolution* **R**.

Proof. The repetition of the above construction will give us the simplicial resolution of an algebra.

Let B be a commutative **k**-algebra. We describe the zero step of the construction. It consists of the choice of a free **k**-algebra E and a surjection $f\colon E \to B$ which gives an isomorphism $E/\mathrm{Ker}\, f \cong B$ as **k**-algebras. Then we form the trivial simplicial algebra $\mathbf{E}^{(0)}$ for which in every degree n, $E_n = E$ and $d_i^n = \mathrm{id} = s_j^n$ for all i, j. Thus $\mathbf{E}^{(0)} = \mathbf{K}(E, 0)$ and $\pi_0(\mathbf{E}^{(0)}) = E$. Now choose a set Ω^0 of generators of the ideal $I = \mathrm{Ker}(E \to^f B)$, and obtain the simplicial algebra in which $E_1^{(1)} = E[\Omega^0]$ and for $n > 1$, $E_n^{(1)}$ is a free E_n-algebra over the degenerate elements. This simplicial algebra is denoted by $\mathbf{E}^{(1)}$ and will be called the 1-*skeleton of the simplicial resolution of an algebra* B.

The consequent steps depend on the choice of sets, $\Omega^0, \Omega^1, \Omega^2, \ldots, \Omega^k, \ldots$ Let $\mathbf{E}^{(k)}$ be the simplicial algebra constructed after k steps, the k-*skeleton of the resolution*. The set Ω^k is formed by elements w of $E_k^{(k)}$ with $d_i^k(w) = 0$ for $0 \le i \le k$ and whose images $\overline{w}$ in $\pi_k(\mathbf{E}^{(k)})$ generate that module over $E_k^{(k)}$.

Finally we have inclusions of simplicial algebras

$$\mathbf{E} = \mathbf{E}^{(0)} \subseteq \mathbf{E}^{(1)} \cdots \subseteq \mathbf{E}^{(k-1)} \subseteq \mathbf{E}^{(k)} \subseteq \cdots,$$

where $\mathbf{E}^{(k+1)} = \mathbf{E}^{(k)}[\Omega^k]$. In passing to the inductive limit (colimit), we obtain an acyclic free simplicial **k**-algebra **R** with $R_n = E_n^{(k)}$ if $n \le k$. **R** is thus a simplicial resolution of the **k**-algebra B. The proof of theorem is completed. ∎

Remark. A variant of the step-by-step construction gives: if **A** is a simplicial algebra, then there exist a free simplicial algebra **E** and an epimorphism $\mathbf{E} \to \mathbf{A}$ which induces isomorphisms on all homotopy modules. (The details are omitted.)

Terminology. The data needed to go from $\mathbf{E}^{(k)}$ to $\mathbf{E}^{(k+1)}$ are precisely a set Ω^k and a function $f^{(k)}\colon \Omega^k \to \mathbf{E}_k^{(k)}$ whose image is contained in $\mathbf{NE}_k^{(k)}$ and which generates $\pi_k(\mathbf{E}^{(k)})$. (We often consider $f^{(k)}$ as being an inclusion and leave it out of the notation.) The pair $(\Omega^k, f^{(k)})$ is then called *k-dimensional construction data for the resolution* and the finite sequence

$$\left((\Omega^0, f^{(0)}), \ldots, (\Omega^{k-1}, f^{(k-1)})\right)$$

is called a kth-*level presentation of the commutative* **k**-*algebra*, B.

The key observation, which follows from the universal property of the polynomial ring construction, is a freeness statement:

PROPOSITION 1.3. *Let* $\mathbf{E}^{(k)}$ *be a k-skeleton of a simplicial resolution of B and* $(\Omega^k, f^{(k)})$ *k-dimensional construction data. Suppose given a simplicial algebra morphism* $\Theta\colon \mathbf{E}^{(k)} \to \mathbf{F}$ *such that* $\Theta_*(\pi_k(\mathbf{E}^{(k)})) = 0$, *then* Θ *extends over* $\mathbf{E}^{(k+1)}$.

This freeness statement does not contain a uniqueness clause. That can be achieved by choosing a lift for $\Theta_k f^{(k)}$ to NF_{k+1}, a lift that must exist since $\Theta_*(\pi_k(\mathbf{E}^{(k)}))$ is trivial.

When handling combinatorially defined resolutions, rather than functorially defined ones, of free algebras on a simplicial set, this proposition is as close to ''left adjointness'' as is possible without entering the realm of homotopical algebra to an extent greater than is desirable for us here.

We will not discuss here homotopics of simplicial algebra morphisms, and so will not discuss homotopy invariance of the above construction, for which see André [1].

1.3. *Crossed Modules*

Whitehead [19] used crossed modules in various contexts, especially in his investigations into the algebraic structure of relative homotopy groups. In this section, we recall the definition and elementary theory of crossed modules of commutative algebras given by the second author [14]. More details about this may be found in [18, 4].

Throughout this paper we denote an action of $r \in R$ on $m \in M$ by $r \cdot m$.

Let R be a **k**-algebra with identity. A *pre-crossed module of commutative algebras* is an R-algebra C, together with an R-algebra morphism.

$$\partial\colon C \to R,$$

such that for all $c \in C, r \in R$

$$\partial(r \cdot c) = r\,\partial c. \qquad \text{(CM1)}$$

This is a *crossed module* if in addition, for all $c, c' \in C$,

$$\partial c \cdot c' = cc'. \tag{CM2}$$

This second condition is called the *Peiffer identity*. We denote such a crossed module by (C, R, ∂). Clearly any crossed module is a pre-crossed module.

A morphism of crossed modules from (C, R, ∂) to (C', R', ∂') is a pair of **k**-algebra morphisms,

$$\theta: C \to C', \qquad \psi: R \to R',$$

such that

$$\theta(r \cdot c) = \psi(r) \cdot \theta(c) \qquad \text{and} \qquad \partial'\theta(c) = \psi\partial(c).$$

In this case, we will say that θ is a crossed R-module morphism if $R = R'$ and ψ is the identity.

LEMMA 1.4. *Assume a simplicial algebra* **E** *and a simplicial ideal* **I** *are given. The inclusion*

$$\text{inc}: \mathbf{I} \hookrightarrow \mathbf{E}$$

induces a map

$$\partial: \pi_0(\mathbf{I}) \to \pi_0(\mathbf{E}),$$

and **E** *acting on* **I** *by multiplication induces an action of* $\pi_0(\mathbf{E})$ *on* $\pi_0(\mathbf{I})$. *Then* $(\pi_0(\mathbf{I}), \pi_0(\mathbf{E}), \partial)$ *is a crossed module.*

Proof. It is straightforward from a direct calculation. ∎

Any crossed module can be obtained as π_0 of an ideal inclusion, $\mathbf{I} \hookrightarrow \mathbf{E}$, of simplicial algebras but we will not include a proof here.

1.4. *Free Crossed Modules*

We will need the notion of a free crossed module of commutative algebras which seems first to have been described by Aznar Garcia [4].

Let (C, R, ∂) be a crossed module, let Y be a set, and let $\nu: Y \to C$ be a function; then (C, R, ∂) is said to be a *free crossed R-module with basis ν* or, alternatively, on the function $\partial\nu: Y \to R$ if for any crossed R-module (C', R, ∂') and function $\nu': Y \to C'$ such that $\partial'\nu' = \partial\nu$, there is a unique morphism

$$\phi: (C, R, \partial) \to (C', R, \partial')$$

such that $\phi\nu = \nu'$.

The crossed module (C, R, ∂) is *totally free* if R is a free algebra. On replacing "crossed" by "pre-crossed" in the above definition of a (totally) free crossed module, we obtain the appropriate definition of a (*totally*) *free pre-crossed module*.

The following result is proved in [14].

THEOREM 1.5. *A free crossed module R-module* (C, R, ∂) *exists on any function* $f: Y \to R$ *with codomain* R.

The ideal of the construction is as follows:

Given a function from a set Y to the **k**-algebra R, $f: Y \to R$, consider $E = R^+[Y]$, the positively graded part of the polynomial ring on Y so that R acts on E by multiplication. The function f induces a morphism of R-algebras

$$\theta: R^+[Y] \to R$$

given by $\theta(y) = f(y)$. Let P be the ideal of $R^+[Y]$ generated by all elements of the form

$$P = \{pq - \theta(p)q : p, q \in R^+[Y]\}.$$

then take $C = E/P$.

Remark. Later on, P will be denoted by P_1 and will be called the first order Peiffer ideal, as our intention is to use higher order versions of the Peiffer elements.

We state relations between free crossed module and the Koszul complex from Porter [14] that were already hinted at in Lichtenbaum and Schlessinger [11].

PROPOSITION 1.6 [14]. *If* (C, R, ∂) *is a free crossed module R-module on a function* $f: Y \to R$, *with* $Y = \{y_1, \ldots, y_n\}$, *then there is a natural isomorphism*

$$C \cong R^n/\mathrm{Im}\, d,$$

where $d: \Lambda^2 R^n \to R^n$ *is the Koszul differential.*

2. HIGHER ORDER PEIFFER ELEMENTS AND CROSSED COMPLEXES

We briefly recall from [3] the following results:

Let **E** be a simplicial commutative algebra with Moore complex **NE** and for $n > 1$, let D_n be the ideal generated by the degenerate elements in

dimension n. If $E_n = D_n$, then

$$\partial_n(NE_n) = \partial_n(I_n) \qquad \text{for all } n > 1,$$

where I_n is an ideal in E_n generated by a fairly small explicitly given set of elements (see below).

If $n = 2$, 3, or 4, then the image of the Moore complex of the simplicial algebra **E** can be given in the form

$$\partial_n(NE_n) = \sum_{I,J} K_I K_J,$$

where $\varnothing \neq I, J \subset [n-1] = \{0, 1, \ldots, n-2\}$ with $I \cup J = [n-1]$, and where

$$K_I = \bigcap_{i \in I} \operatorname{Ker} d_i \qquad \text{and} \qquad K_J = \bigcap_{j \in J} \operatorname{Ker} d_j.$$

In general for $n > 4$, there is an inclusion

$$\sum_{I,J} K_I K_J \subset \partial_n(NE_n).$$

Let $S(n, n-r)$ be the set of all monotone increasing surjective maps from $[n]$ to $[n-r]$. This can be generated from the various σ_i^n by composition. The composition of these generating maps is subject to the rule $\sigma_j \sigma_i = \sigma_{i-1}\sigma_j$, $j < i$. This implies that every element $\sigma \in S(n, n-r)$ has unique expression as $\sigma = \sigma_{i_1} \circ \sigma_{i_2} \circ \cdots \circ \sigma_{i_r}$ with $0 \le i_1 < i_2 < \cdots < i_r \le n-1$, where the indices i_k are the elements of $[n]$ such that $\{i_1, \ldots, i_r\} = \{i : \sigma(i) = \sigma(i+1)\}$. We thus can identify $S(n, n-r)$ with the set $\{(i_r, \ldots, i_1) : 0 \le i_1 < i_2 < \cdots < i_r \le n-1\}$. In particular, the single element of $S(n, n)$, defined by the identity map on $[n]$, corresponds to the empty 0-tuple ( ) denoted by $\varnothing_n$. Similarly the only element of $S(n, 0)$ is $(n-1, n-2, \ldots, 0)$. For all $n \ge 0$, let

$$S(n) = \bigcup_{0 \le r \le n} S(n, n-r).$$

Let $P(n)$ be a set consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \varnothing$, where $\alpha = (i_r, \ldots, i_1)$, $\beta = (j_s, \ldots, j_1) \in S(n)$. We write $\#\alpha = r$, i.e., the length of the string α. The **k**-linear morphisms that we will need,

$$\{C_{\alpha,\beta} : NE_{n-\#\alpha} \otimes NE_{n-\#\beta} \to NE_n : (\alpha, \beta) \in P(n),\ n \ge 0\},$$

are given as composites $C_{\alpha,\beta} = p\mu(s_\alpha \otimes s_\beta)$, where

$$s_\alpha = s_{i_r} \dots s_{i_1} \colon NE_{n-\#\alpha} \to E_n, \qquad s_\beta = s_{j_s} \dots s_{j_1} \colon NE_{n-\#\beta} \to E_n,$$

$p \colon E_n \to NE_n$ is defined by composite projections $p = p_{n-1} \dots p_0$, where $p_j = 1 - s_j d_j$ with $j = 0, 1, \dots, n-1$, and we denote the multiplication by $\mu \colon E_n \otimes E_n \to E_n$. Thus

$$C_{\alpha,\beta}(x_\alpha \otimes y_\beta) = (1 - s_{n-1}d_{n-1}) \dots (1 - s_0 d_0)\big(s_\alpha(x_\alpha) s_\beta(y_\beta)\big).$$

Define the ideal I_n to be that generated by elements of the form $C_{\alpha,\beta}(x_\alpha \otimes y_\beta)$, where $x_\alpha \in NE_{n-\#\alpha}$ and $y_\beta \in NE_{n-\#\beta}$.

The idea for the construction of I_n and the use of the structure maps came from examining the thesis of Carrasco [6]; see also Carrasco and Cegarra [7].

The final elements that we need are the definition of a crossed complex of algebras, and a construction of a crossed complex from a simplicial algebra. The proof that this works uses these $C_{\alpha,\beta}$ maps in a neat way.

A crossed complex of **k***-algebras* is a sequence of **k**-algebras

$$\mathscr{C}: \qquad \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0,$$

in which

(i) (C_1, C_0, ∂_1) is a crossed module,

(ii) for $i > 1$, C_i is an C_0-module on which $\partial_1 C_1$ operates trivially and each ∂_i is an C_0-module morphism,

(iii) for $i \geq 1$, $\partial_{i+1}\partial_i = 0$.

Morphisms of crossed complexes are defined in the obvious way.

The homology of a crossed complex $\mathscr{C}$ can be defined by

$$H_n(\mathscr{C}) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}.$$

A crossed complex $\mathscr{C}$ of **k**-algebras is *exact* if for $n \geq 1$,

$$\operatorname{Ker}(\partial_n \colon C_n \to C_{n-1}) = \operatorname{Im}(\partial_{n+1} \colon C_{n+1} \to C_n).$$

A crossed resolution of a **k***-algebra* B is a crossed complex

$$\mathscr{C}: \qquad \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

of **k**-algebras, where ∂_1 is a crossed C_0-module, together with $f \colon C_0 \to B$ a morphism, such that the sequence

$$\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{f} B \to 0$$

is exact.

If, for $i \geq 0$, the C_i are free and ∂_1 is a free crossed module, then the resolution is called a *free crossed resolution* of the **k**-algebra B.

The "step-by-step" construction of a crossed resolution is analogous to the construction of a (chain complex) resolution of B as a module. Pick a free **k**-algebra A with an epimorphism $f\colon A \to B$, then pick generators for the ideal, $\operatorname{Ker} f$. Use these generators to form a free crossed module, whose kernel will be a B-module. Finally, resolve that B-module in your favorite way and put that resolution in its place. The result is the desired crossed resolution of B.

The use of crossed resolutions and more generally of crossed complexes in combinatorial and cohomological algebra theory has the advantage of being less cumbersome than the full simplicial theory, but certain structural invariants are lost when they are used, as such crossed resolutions do not represent all the possible homotopy types available. It is therefore important to be able to go from the simplicial context to the crossed one and to study what is lost in the process.

For a simplicial group **G**, Carrasco and Cegarra [7] defined

$$C_n(\mathbf{G}) = \frac{NG_n}{(NG_n \cap D_n)\, d_{n+1}(NG_{n+1} \cap D_{n+1})}.$$

This constructs a crossed complex of groups from the Moore complex of **G**. Their proof requires an understanding of hypercrossed complexes. Ehlers and Porter [9] developed a direct proof for simplicial groups/groupoids independently of [7]. Here we will carry out a short proof for the algebra case by using the higher order Peiffer elements and will show that $C(\mathbf{E})$ is a crossed complex where we write

$$C_n(\mathbf{E}) = \frac{NE_n}{(NE_n \cap D_n) + d_{n+1}(NE_{n+1} \cap D_{n+1})}.$$

If $x \in NE_n$, we will write $\bar{x}$ for the corresponding element of $C_n(\mathbf{E})$. The map $\partial_n\colon C_n(\mathbf{E}) \to C_{n-1}(\mathbf{E})$ will be that induced by d_n^n.

We first check that the quotient algebra does exist.

LEMMA 2.1. *The subalgebra* $(NE_n \cap D_n) + d_{n+1}(NE_{n+1} \cap D_{n+1})$ *is an ideal in* E_n.

Proof. For any $a \in NE_n \cap D_n$, $x \in NE_{n+1} \cap D_{n+1}$, and $z \in E_n$, the element $z(a + d_{n+1}x)$ can be written in the form

$$z(a + d_{n+1}x) = s_{n-1}d_n(z)a + d_{n+1}(s_n(z)x + s_n z s_n a - s_{n-1}z s_n a)$$

and so is in $(NE_n \cap D_n) + d_{n+1}(NE_{n+1} \cap D_{n+1})$. ∎

PROPOSITION 2.2. *Let E be a simplicial algebra; then defining*

$$C_n(\mathbf{E}) = \frac{NE_n}{(NE_n \cap D_n) + d_{n+1}(NE_{n+1} \cap D_{n+1})}$$

with

$$\partial_n(\bar{z}) = \overline{d_n z}$$

yields a crossed complex $C(\mathbf{E})$ of algebras.

Proof. (i) Axiom follows since

$$C_1(\mathbf{E}) = \frac{NE_1}{\partial_2(NE_2 \cap D_2)} = \frac{NE_1}{\operatorname{Ker} d_0 \operatorname{Ker} d_1},$$

and in [3], we saw that $\operatorname{Ker} d_0 \operatorname{Ker} d_1$ contains the Peiffer elements so $(C_1(\mathbf{E}), C_0(\mathbf{E}), \partial)$ is a crossed module.

(ii) Consider the generators $C_{\alpha,\beta}(x_\alpha \otimes y_\beta)$ of the ideal $NE_{n+1} \cap D_{n+1}$. For $x \in NE_n$ and $y \in NE_r$, by taking $\alpha = (n, n-1, \ldots, r)$, $\beta = (r-1)$, it is easy to see that

$$C_{(n,n-1,\ldots,r),(r-1)}(x \otimes y) = s_n \ldots s_r(x)\left(\sum_{k=0}^{n-r+1} (-1)^k s_{r-1+k}(y)\right)$$

and then

$$d_{n+1}C_{\alpha,\beta}(x \otimes y) = s_{n-1} \ldots s_r(x)\left(\sum_{k=0}^{n-r} (-1)^k s_{r-1+k} d_n(y) + (-1)^{n-r+1} y\right)$$

$$= s_r^{(n-r)}(x)\left(\sum_{k=0}^{n-r} (-1)^k s_{r-1+k} d_n(y) + (-1)^{n-r+1} y\right).$$

This implies that

$$s_r^{(n-r)}(x)y \in \left[NE_n \cap D_n + d_{n+1}(NE_{n+1} \cap D_{n+1})\right],$$

which shows that the actions of NE_r on NE_n defined by multiplication

$$x \cdot y = s_r^{(n-r)}(x)y$$

via degeneracies are trivial if $r \geq 1$. For $r = 1$, this gives $\alpha = (n, n-1, \ldots, 1)$, $\beta = (0)$, and

$$C_{(n,n-1,\ldots,1)(0)}(x \otimes y) = s_n s_{n-1} \ldots s_1(x)\big(s_0 y - s_1 y + \cdots + (-1)^n s_n y\big),$$

where $x \in NE_1$, $y \in NE_n$, and it is easily checked that

$$d_{n+1} C_{\alpha,\beta}(x \otimes y) = s_{n-1} \ldots s_1(x) y + s_{n-1} \ldots s_1(x) s_0 d_n y.$$

Then

$$s_{n-1} \ldots s_1(x) y \equiv 0 \qquad \mod \big[NE_n \cap D_n + d_{n+1}(NE_{n+1} \cap D_{n+1})\big].$$

This gives the following: if $\bar{x} \in C_1$ then $\bar{x}$ and $\partial_1 \bar{x}$ act on C_n in the same way, and so $\partial_1 C_1$ acts trivially on C_n.

(iii) By defining

$$\partial_n(\bar{z}) = \overline{d_n^n(z)} \qquad \text{with } z \in NE_n,$$

one obtains a well defined map $\partial\colon C_n(\mathbf{E}) \to C_{n-1}(\mathbf{E})$ satisfying $\partial\partial = 0$. ∎

3. EXAMPLES OF "STEP-BY-STEP" CONSTRUCTIONS AND CROSSED COMPLEXES IN LOW DIMENSIONS

In this section, we describe in more detail 2-dimensional data and thus the "step-by-step" construction of a free simplicial algebra and its skeleton up to dimension 2. We will interpret this construction and see how it relates to other algebraic constructions such as those of free crossed modules and Koszul complexes.

Remark. In our examples we will assume that rings and algebras are finitely generated, in fact Noetherian. This is only for ease of exposition.

First, we need a general result which is well known:

Let A be a subring of a commutative ring S, and consider the polynomial ring $A[X_1, \ldots, X_n]$ over A in n indeterminates $X_1, \ldots, X_n$. Let $a_1, \ldots, a_n \in S$. There is exactly one ring homomorphism $g\colon A[X_1, \ldots, X_n] \to S$ with the properties that

$$g(r) = r \qquad \text{and} \qquad g(X_i) = a_i \qquad \text{for all } i = 1, \ldots, n \text{ and } r \in A.$$

This homomorphism g is called the *evaluation homomorphism* or just evaluation at $a_1, \ldots, a_n$.

If $g\colon A[X_1, \ldots, X_n] \to A$ is the evaluation homomorphism at $a_1, \ldots, a_n$, then

$$\operatorname{Ker} g = (X_1 - a_1, \ldots, X_n - a_n).$$

Let R be a commutative $\mathbf{k}$-algebra with an ideal $I = (x_1, \ldots, x_n)$ of R generated by the elements $x_1, \ldots, x_n$ in R. Let $\mathbf{K}(R, 0)$ denote the simplicial algebra which in every dimension is equal to R and $d_i = \mathrm{id} = s_j$, for all i, j.

There is an obvious epimorphism, $f\colon R \to R/(x_1, \ldots, x_n)$, which gives an isomorphism $R/\mathrm{Ker} f \cong B$, where $B = R/I$.

Let

$$\Omega^0 = \{x_1, \ldots, x_n\} \subset \mathrm{Ker}\, f.$$

(For the construction of a resolution, Ω^0 is chosen to generate $\mathrm{Ker}\, f$ as an ideal.) The 1-skeleton $\mathbf{E}^{(1)}$ of the free simplicial resolution of B can be built by adding new indeterminates $X = \{X_1, \ldots, X_n\}$ into $E_1^{(0)} = R$ to form $E_1^{(1)} = E_1^{(0)}[X] = R[X_1, \ldots, X_n]$, with the face maps and degeneracy map

$$R[X_1, \ldots, X_n] \underset{s_0}{\overset{d_0, d_1}{\rightleftarrows}} R$$

given by $d_1^1(X_i) = x_i \in \mathrm{Ker}\, f$, $d_0^1(X_i) = 0$, and $s_0(r) = r \in R$. Thus the augmented 1-skeleton $\mathbf{E}^{(1)}$ looks like

$$\mathbf{E}^{(1)}\colon \qquad \ldots R[s_0(X), s_1(X)] \underset{s_0, s_1}{\overset{d_0, d_1, d_2}{\rightleftarrows}} R[X] \underset{s_0}{\overset{d_0, d_1}{\rightleftarrows}} R \xrightarrow{f} R/I.$$

Note that for $n > 1$ higher levels of $\mathbf{E}^{(1)}$ are generated by the degenerate elements.

LEMMA 3.1. *We assume the* 1*-skeleton* $\mathbf{E}^{(1)}$, *is given, as above. Then*
(i) $\mathrm{Ker}\, d_0^1 = R^+[X_1, \ldots, X_n] = (X_1, \ldots, X_n)$,
(ii) $\mathrm{Ker}\, d_1^1 = (X_1 - x_1, \ldots, X_n - x_n)$.

Proof. Clear. ∎

Note that $\pi_0(\mathbf{E}^{(1)}) \cong B$.

For any simplicial algebra $\mathbf{E}$, if $\mathbf{E}$ equals its 1-skeleton, so that $\mathbf{E} = \mathbf{E}^{(1)}$, then

$$\pi_1(\mathbf{E}) = \mathrm{Ker}\big(d_1^1\colon \mathrm{Ker}\, d_0^1/\mathrm{Ker}\, d_0^1 \mathrm{Ker}\, d_1^1 \to E_0\big).$$

Indeed, by definition, the first homotopy module looks like

$$\pi_1(\mathbf{E}) = \big(\mathrm{Ker}\, d_0^1 \cap \mathrm{Ker}\, d_1^1\big)/d_2^2\big(\mathrm{Ker}\, d_0^2 \cap \mathrm{Ker}\, d_1^2\big).$$

The denominator of this homotopy module is exactly

$$\partial_2(NE_2) = d_2^2\big(\operatorname{Ker} d_0^2 \cap \operatorname{Ker} d_1^2\big) = \operatorname{Ker} d_0^1 \operatorname{Ker} d_1^1.$$

Consider the morphism $\delta\colon \operatorname{Ker} d_0^1/\partial_2(NE_2) \to E_0$, where δ is induced by d_1. This is a crossed module: NE_0 acts on $NE_1/\partial_2 NE_2$ by multiplication via s_0, i.e.,

$$\bar{x} \cdot y = \overline{s_0(y)x},$$

where for $x \in NE_1$, $\bar{x}$ denotes the corresponding element of $NE_1/\partial_2 NE_2$. Since for $x, y \in NE_1$, $x(s_0 d_1 y) - xy = x(s_0 d_1 y - y) \in \operatorname{Ker} d_0 \operatorname{Ker} d_1 = \partial_2 NE_2$, one can readily check that δ is a crossed module.

Finally, using $\operatorname{Ker}(d_1\colon \operatorname{Ker} d_0 \to E_0) = \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1$, one obtains

$$\pi_1(\mathbf{E}) = \operatorname{Ker}(NE_1/\operatorname{Ker} d_0 \operatorname{Ker} d_1 \to E_0).$$

In general, for $k \geq 1$ we have that if $\mathbf{E}^{(k)}$ is the **k**-skeleton of the free simplicial algebra, **E**, then

$$\pi_k(\mathbf{E}^{(k)}) = \operatorname{Ker}\big(NE_k^{(k)}/\partial_{k+1}(NE_{k+1}) \to E_{k-1}\big).$$

PROPOSITION 3.2. *For any simplicial algebra* **E** *that satisfies* $E_k = D_k$ *for* $k \geq 2$ (*i.e., no non-degenerate generators above dimension* 1), *then* $\partial_2(NE_2)$ *is generated by Peiffer elements. In particular if* $\mathbf{E} = \mathbf{E}^{(1)}$ *as constructed above,* $\partial_2 NE_2$ *is generated by the elements* $(X_i - x_i)X_j$, *for* X_i, X_j *among the generators of* E_1 *as a polynomial algebra over* E_0.

Proof. By the discussion in Section 2 for the case $n = 2$, we have $\partial_2(NE_2) = \operatorname{Ker} d_0^1 \operatorname{Ker} d_1^1$ and if $a \in \operatorname{Ker} d_0^1, b \in \operatorname{Ker} d_1^1$, then $b = b' - s_0 d_1 b'$ for some $b' \in \operatorname{Ker} d_0^1$, in fact $b' = b - s_0 d_0 b'$. Hence

$$ab = ab' - as_0 d_1 b' = b'a - \partial b' \cdot a,$$

which is a Peiffer element.

If $\mathbf{E} = \mathbf{E}^1$ is the 1-skeleton of a free resolution, then $\operatorname{Ker} d_0^1 \operatorname{Ker} d_1^1$ is an ideal corresponding to pairs of generating elements of the form $(X_i - x_i)X_j$ with $1 \leq i, j \leq n$ and these are exactly the Peiffer elements. ∎

PROPOSITION 3.3. *Given a presentation* $P = (R; x_1, \ldots, x_n)$ *of an R-algebra B and* $\mathbf{E}^{(1)}$ *the* 1*-skeleton of the free simplicial algebra generated by this presentation, then*

$$\delta\colon NE_1^{(1)}/\partial_2\big(NE_2^{(1)}\big) \to NE_0^{(1)}$$

is the free crossed module on $(x_1, \ldots, x_n) \to R$. *In particular,*

$$\pi_1(\mathbf{E}^{(1)}) \cong \mathrm{Ker}(C \to R),$$

where $C \cong R^n/\mathrm{Im}\, d$, *for* d: $\Lambda^2 R^n \to R$, *the second Kozsul differential.*

Proof. As we noted earlier, there is an equality

$$\pi_1(\mathbf{E}^{(1)}) = \mathrm{Ker}(NE_1/\mathrm{Ker}\, d_0 \mathrm{Ker}\, d_1 \to E_0).$$

It follows from Lemma 3.1 that $NE_1^{(1)} = \mathrm{Ker}\, d_0^1 = R^+[X_1, \ldots, X_n]$. Moreover by the previous proposition, $\partial_2(NE_2^{(1)}) = \mathrm{Ker}\, d_0 \mathrm{Ker}\, d_1$ is generated by the Peiffer elements of the form $(X_i - x_i)X_j$ with $1 \le i, j \le n$. From Theorem 1.5, we can thus define a free crossed module

$$\delta\colon R^+[X_1, \ldots, X_n]/\mathrm{Ker}\, d_0^1 \mathrm{Ker}\, d_1^1 \to R.$$

Repeating the argument from [14], any polynomial in $R^+[X_1, \ldots, X_n]$ is congruent modulo $\mathrm{Ker}\, d_0^1 \mathrm{Ker}\, d_1^1$ to a monomial, i.e., an element in $R^{\{X_1, \ldots, X_n\}}$, the free module R^n with basis $X_1, \ldots, X_n$. This module has an algebra structure given by $X_i X_j \equiv x_i X_j \equiv x_j X_i \bmod P_1$. We put

$$C \cong R^+[X_1, \ldots, X_n]/\mathrm{Ker}\, d_0^1 \mathrm{Ker}\, d_1^1$$

and apply Proposition 1.6, which gives $C = R^n/\mathrm{Im}\, d$, where d: $\Lambda^2 R^n \to R^n$ is the usual Koszul differential carrying the element $X_i \wedge X_j$ to $\varphi(X_i)X_j - X_i\varphi(X_j)$, φ: $R^n \to R$ being given by $\varphi(X_i) = x_i$. ∎

Thus 3.1 gives the following corollary:

COROLLARY 3.4.

$$\pi_1(\mathbf{E}^{(1)}) = \mathrm{Ker}(R^n/\mathrm{Im}\, d \to R).$$

We now will return to the next step of the construction of a free simplicial algebra. First, we took a set of generators

$$\Omega^1 = \{y_1, \ldots, y_m\} \subset \pi_1(\mathbf{E}^{(1)})$$

and killed off the elements in the homotopy module $\pi_1(\mathbf{E}^{(1)})$ by adding new indeterminates $Y = \{Y_1, \ldots, Y_m\}$ into $E_2^{(1)}$ to establish

$$E_2^{(2)} = E_2^{(1)}[Y] = \big(R[s_0(X), s_1(X)]\big)[Y].$$

together with

$$d_0^2(Y_i) = 0, \qquad d_1^2(Y_i) = 0, \qquad d_2^2(Y_i) = y_i.$$

Hence the augmented 2-skeleton $\mathbf{E}^{(2)}$ looks like

$$\mathbf{E}^{(2)}: \qquad \ldots \big(R\big[s_0(X), s_1(X)\big]\big)[Y] \underset{s_0, s_1}{\overset{d_0, d_1, d_2}{\rightleftarrows}} R[X] \underset{s_0}{\overset{d_0, d_1}{\rightleftarrows}} R \xrightarrow{f} R/I.$$

For $\mathbf{E}^{(2)}$, levels higher than dimension 2 are generated by degeneracy elements. In the next section we will examine the second level of the corresponding crossed complex.

4. FREE CROSSED RESOLUTIONS

A ''step-by-step'' construction of a free simplicial algebra is constructed from simplicial algebra inclusions

$$\mathbf{E}^{(0)} \subset \mathbf{E}^{(1)} \subset \mathbf{E}^{(2)} \subset \ldots.$$

In the following, we take the functor **C**, which is described in Section 2, to see what $C(\mathbf{E}^{(k)})$ looks like, where $\mathbf{E}^{(k)}$ is the **k**-skeleton of that construction.

Recall the ''step-by-step'' construction of the free augmented simplicial algebra **E**,

$$\mathbf{E}^{(0)}: \qquad \ldots R \to R \to R/(x_1, \ldots, x_n),$$

so $\mathbf{E}^{(0)}$ is the trivial simplicial algebra in which in every degree n, $E_n^{(0)} = R$ and $d_i^n = \mathrm{id} = s_j^n$. It is easy to see that $C_0(\mathbf{E}^{(0)}) = R$ as $NE_1 \cap D_1$ is trivial. As $E_2^{(1)}$ is generated by the degenerate elements, $E_2^{(1)} = D_2$, so the crossed complex term $C_1(\mathbf{E}^{(1)})$ is

$$\begin{aligned} C_1(\mathbf{E}^{(1)}) &= \frac{NE_1^{(1)}}{\big[NE_1^{(1)} \cap D_1 + \partial_2\big(NE_2^{(1)} \cap D_2\big)\big]} \\ &= \frac{NE_1^{(1)}}{\partial_2\big(NE_2^{(1)} \cap D_2\big)} \qquad \text{since } NE_1 \cap D_1 = 0, \\ &= \frac{NE_1^{(1)}}{\partial_2\big(NE_2^{(1)}\big)} \qquad \text{as } E_2^{(1)} = D_2. \end{aligned}$$

By Lemma 3.1 and Proposition 3.2, we have $NE_1^{(1)} = R^+[X_1, \ldots, X_n]$ and $\partial_2(NE_2^{(1)})$ is generated by the Peiffer elements, respectively. It then follows

that

$$C_1(\mathbf{E}^{(1)}) = R^+[X_1, \ldots, X_n]/P_1.$$

Here P_1 is the first order Peiffer ideal. The proof of Theorem 1.5 shows that

$$\partial_1 : R^+[X_1, \ldots, X_n]/P_1 \to R$$

is a free crossed module.

We next look at the 2-skeleton of the construction. As before, $E_3^{(2)} = D_3$ as $E_3^{(2)}$ is generated by the degenerate elements. Thus the second term of the crossed complex is

$$C_2(\mathbf{E}^{(2)}) = \frac{NE_2^{(2)}}{\left[NE_2^{(2)} \cap D_2 + \partial_3\left(NE_3^{(2)} \cap D_3\right)\right]},$$

$$= \frac{NE_2^{(2)}}{\left[NE_2^{(2)} \cap D_2 + \partial_3\left(NE_3^{(2)}\right)\right]} \qquad \text{as } E_3^{(2)} = D_3.$$

The intersection $NE_2 \cap D_2$ is generated by the elements of the form $s_1x(s_0y - s_1y)$ for $x, y \in NE_1$ and in general, if $x, y \in NE_{n-1}$, then $s_{n-1}x(s_{n-2}y - s_{n-1}y) \in NE_n \cap D_n$. For the case of $\mathbf{E}^{(2)}$, if X_i and X_j are indeterminates used in the construction of $E_1^{(2)}$ then they are in $NE_1^{(2)}$, and the generators of the ideal $NE_2^{(2)} \cap D_2$ are of the form

$$s_1X_i(s_0X_j - s_1X_j).$$

Now look at $\partial_3(NE_3^{(2)})$ in terms of the skeleton $\mathbf{E}^{(2)}$. In a way similar to the proof of Lemma 3.1 and since $d_0^2(Y_i) = d_1^2(Y_i) = 0$, we obtained

$$NE_2^{(2)} = \left(R[s_0(X), s_1(X)]\right)^+[Y] + (s_0(X) - s_1(X)).$$

On the other hand, [3] shows that

$$\partial_3\left(NE_3^{(2)}\right) = \sum_{\{I,J\}} K_1K_J + K_{\{0,1\}}K_{\{0,2\}} + K_{\{0,2\}}K_{\{1,2\}} + K_{\{0,1\}}K_{\{1,2\}},$$

where $I \cup J = [2]$, $I \cap J = \varnothing$, so this is generated by the following elements, for $X_i \in NE_1 = \operatorname{Ker} d_0$ and $Y_i \in NE_2 = \operatorname{Ker} d_0 \cap \operatorname{Ker} d_1$ and for Y_i and $Y_j \in NE_2$ with $1 \le i, j \le n$:

$$(s_1s_0d_1X_i - s_0X_i)Y_j \qquad \text{(i)}$$

$$Y_i(s_1d_2Y_j - Y_j) \qquad \text{(ii)}$$

$$(s_0X_i - s_1X_i)(s_1d_2Y_j - Y_j) \qquad \text{(iii)}$$

$$Y_i(Y_j + s_0d_2Y_j - s_1d_2Y_j) \qquad \text{(iv)}$$

$$s_1X_i(s_0d_2Y_j - s_1d_2Y_j + Y_j) \qquad \text{(v)}$$

$$(s_0d_2Y_i - s_1d_2Y_i + Y_i)(s_1d_2Y_j - Y_j) \qquad \text{(vi)}.$$

The ideal generated by these elements will be denoted by P_2 and will be called the second order Peiffer ideal.

To summarise:

PROPOSITION 4.1. *For a simplicial algebra* $\mathbf{E}$, *if* $\mathbf{E} = \mathbf{E}^{(2)}$, *then the image of the third term of the Moore complex of* $\mathbf{E}^{(2)}$ *is generated by the second order Peiffer elements* P_2.

Finally, writing $Q_2 = NE_2^{(2)} \cap D_2$, we get the second term of the crossed complex as

$$C_2(\mathbf{E}^{(2)}) = \frac{\big(R[s_0(X), s_1(X)]\big)^+[Y] + (s_0(X) - s_1(X)}{Q_2 + P_2}.$$

We thus can form:

PROPOSITION 4.2. *Let* $\mathbf{E}^{(2)}$ *be the* 2*-skeleton of a free simplicial algebra on* 2*-dimensional construction data*. *Then*

$$\mathscr{C}^{(2)}: \qquad \big(R[s_0(X), s_1(X)]\big)^+[Y]/[Q_2 + P_2] \xrightarrow{\partial_2} R^+[X]/P_1 \xrightarrow{\partial_1} R$$

is the 2*-skeleton of a free crossed resolution of* $R/(x_1, \ldots, x_n)$, *where* ∂_2 *and* ∂_1 *are given respectively by*

$$\partial_2[Y_i + (Q_1 + P_2)] = d_2(Y_i) + P_1 \qquad \text{and} \qquad \partial_1(X_i + P_1) = d_1(X_i),$$

for $Y_i \in (R[s_0(X), s_1(X)])^+[Y]$ *and* $X_i \in R[X]^+$.

Proof. This follows immediately from the description of the step-by-step construction of the free simplicial algebra. ▮

We will next turn to the analysis of $C_2(\mathbf{E}^{(2)})$ analogous to that given in [14] for $C_1(E^{(1)})$ (see above).

First, we note that since $Y_i(s_1d_2Y_j - Y_j) \in P_2$, any polynomial in $(R[s_0(X), s_1(X)])^+[Y]$ reduces mod P_2 to a monomial; i.e., the evident module morphism

$$R[s_0(X), s_1(X)]^{(Y)} \to C_2(\mathbf{E}^{(2)})$$

is onto. Turning to an analysis of the module on the left, we note that $R[X]$ acts on it in two ways, via $s_0(X)$ and via $s_1(X)$, but that modulo $Q_2 + P_2$ these two actions coincide. This is an immediate consequence of the crossed complex constructor, $\mathbf{C}$, but can be seen directly as

$$(s_0(X) - s_1(X))Y \equiv (s_0(X) - s_1(X))s_1d_2(Y) \qquad \text{mod } Q_2 + P_2$$

by (iii) above, but the right hand side of this is in $NE_2^{(2)} \cap D_2 = Q_2$ so is zero mod $Q_2 + P_2$. Thus the morphism induced by s_0 from $R[X]$ to $R[s_0(X), s_1(X)]$ induces an epimorphism

$$R[X]^{(Y)} \to C_2(\mathbf{E}^{(2)}).$$

The action of X_i on Y_j is via $d_1 X_i = x_i$ by (ii), but also by (v),

$$X_i \cdot Y_j = s_1(X_i)Y_j \equiv s_1(X_i)(s_1 d_2 Y_j - s_0 d_2 Y_j) \equiv 0 \qquad \text{mod } Q_2 + P_2,$$

so we conclude that

$$x_i \cdot Y_j \equiv 0 \qquad \text{mod } Q_2 + P_2$$

and so we have an epimorphism,

$$\varphi\colon B^{(Y)} \to C_2(\mathbf{E}^{(2)}),$$

of B-modules. This epimorphism is defined by, for each $b \in B$, picking an element $p_b \in R$ mapping down to b, then $\varphi(bY_i) = p_b Y_i + (Q_2 + P_2)$ and this is independent of the choice of p_b.

This epimorphism is an isomorphism. To show this we can adopt a method that works in more generality. We shall show that $C_2(\mathbf{E}^{(2)})$ is free on Y as a B-module. Let M be any B-module and suppose we are given an assignment,

$$\Theta\colon Y \to M,$$

of elements of M to elements of Y. Now form a crossed complex having B in dimension 0 and M in dimension 2,

$$\to 0 \to M \to 0 \to B,$$

with the action of B on M being the given one.

We form a simplicial algebra $S(M, B)$ with Moore complex exactly this crossed complex. (No knowledge of the reconstruction in general of simplicial algebras from hypercrossed complexes is needed here as the classical Dold–Kan theorem for simplicial abelian groups gives the additive structure while the only non-trivial multiplications come from the "semidirect" multiplications on terms of the form $M \oplus B$.) In low dimensions this is

$$S(M, B)_0 = S(M, B)_1 = B,$$
$$S(M, B)_2 = M \oplus B,$$
$$S(M, B)_3 = s_0(M) \oplus s_1(M) \oplus B,$$

with fairly obvious or trivial morphisms as face and degeneracy morphisms.

We look again at $\mathbf{E}^{(1)}$ (for instance, before Lemma 3.1). There is an obvious morphism φ from $\mathbf{E}^{(1)}$ to $S(M, B)$ induced by the quotient morphism $f\colon R \to R/I \cong B$. As $\pi_1(S(M, B))$ is zero, Proposition 1.3 applies to show that φ extends uniquely over $\mathbf{E}^{(2)}$ extending the assignment Θ.

Passing back via $\mathbf{C}$ to crossed complexes we get $C(\varphi)$ maps $C_2(\mathbf{E}^{(2)})$ to M extending Θ in a unique way. We have thus proved that $C_2(\mathbf{E}^{(2)})$ is free on Y. It is now a simple matter to identify the isomorphism as that given earlier.

Of course the above argument generalises easily to any dimension above 2 as well so we get

PROPOSITION 4.3. *IF* $\mathbf{E}$ *is a simplicial resolution of B given by a construction data sequence* $\{(\Omega^{(i)}, f^i), i = 0, 1, \ldots\}$ *and* $\mathbf{E}^{(k)}$ *is the corresponding k-skeleton, then if* $k \geq 2$, $C_k(\mathbf{E}) = C_k(\mathbf{E}^{(k)})$ *is a free B-module on* $\Omega^{(k)}$.

As the homotopy of $\mathbf{E}$ is the homology of $\mathbf{NE}$ and the homology of $\mathbf{NE}$ is the same as that of $C(\mathbf{E})$, it is clear that $C(\mathbf{E})$ is a resolution.

To sum up we have proved:

THEOREM 4.4. *The ''step-by-step'' construction of a simplicial resolution of an algebra, B, yields a ''step-by-step'' construction of a crossed resolution of B via the crossed complex construction,* $\mathbf{C}$.

As a bonus for our method we also have given an explicit description of the crossed complex construction in low dimensions. By analogy with the case of crossed complexes in group cohomology, this should potentially give combinatorial descriptions of a class of non-abelian cocyles and, by default, their ''abelian'' analogues.

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